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A GUIDE TO

MAKING

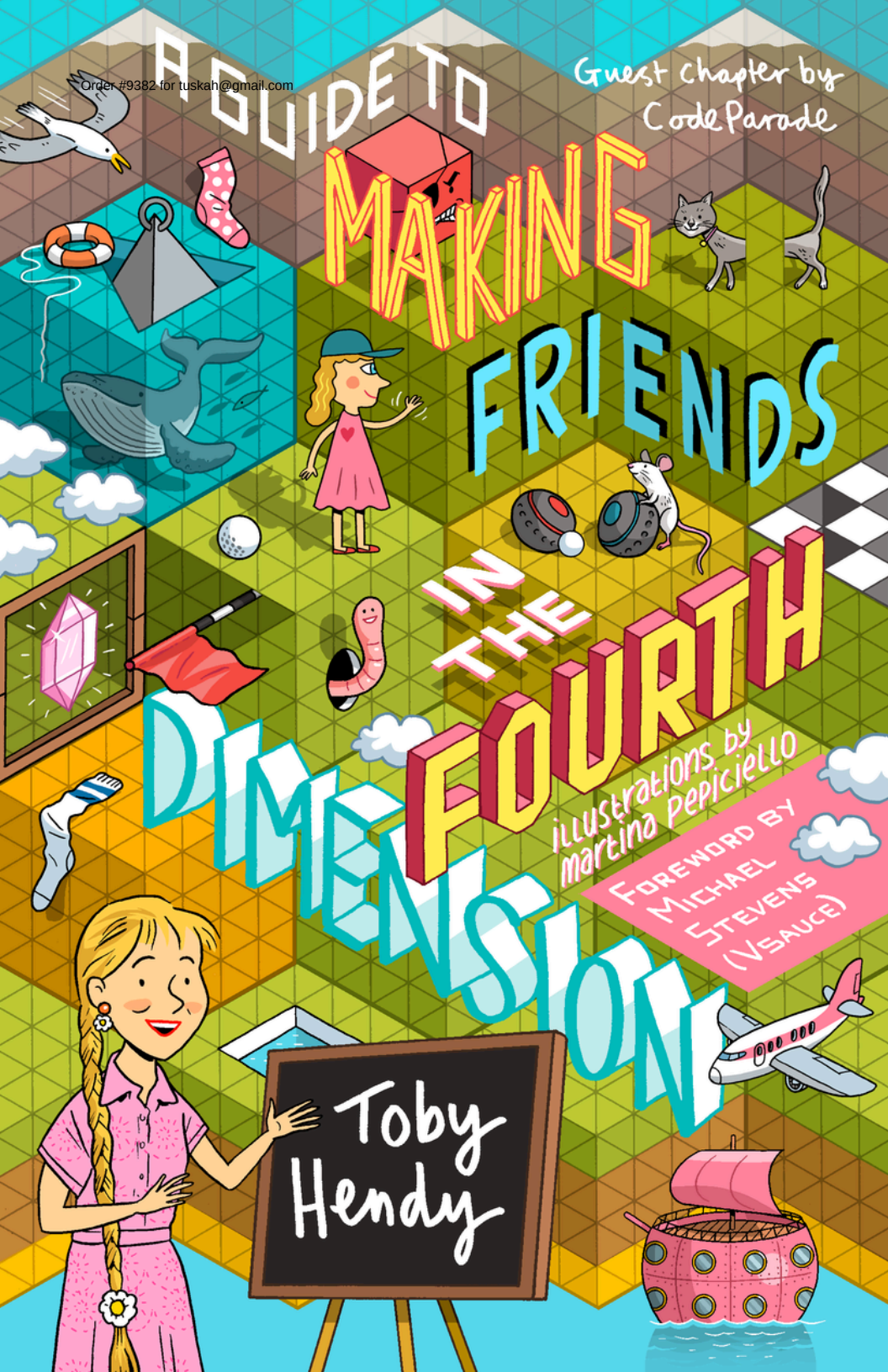
FRIENDS

IN THE
DIMENSION

illustrations by
martina pepiciello

FOREWORD BY
MICHAEL
STEVENS
(VSAUCE)

Toby
Hendy



A GUIDE TO MAKING FRIENDS IN THE FOURTH DIMENSION

By Toby Hendy

With illustrations by Martina Pepiciello

Foreword by Michael Stevens

About the Author

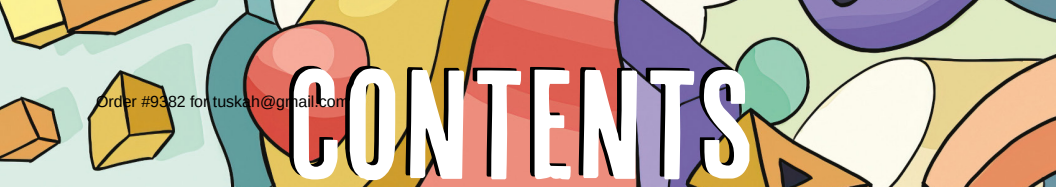


Photo by Josephine Carter

Toby Hendy is a science communicator who makes videos about science and math on her YouTube channel, *Tibeas*, where she has gained an audience of over 1.3 million subscribers. *A Guide to Making Friends in the Fourth Dimension* is her first book.

To my parents





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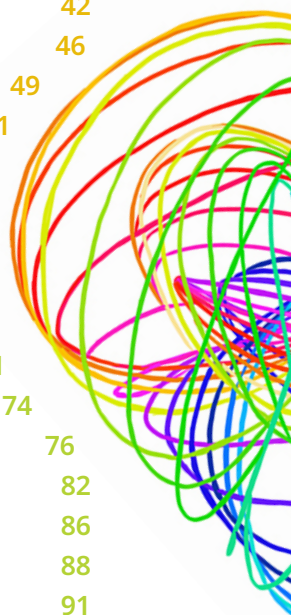
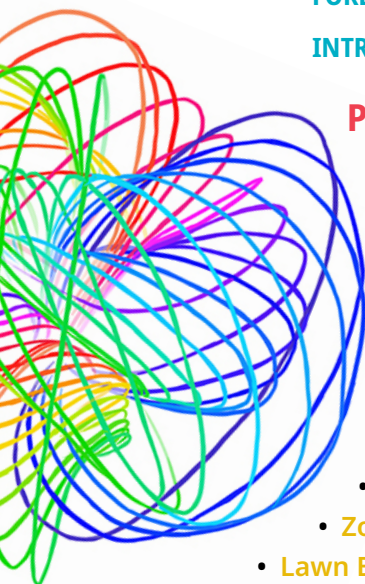
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FOREWORD BY MICHAEL STEVENS

Seven years ago, deep in Peruvian Amazonia, I sat on a thin mattress in a large hut surrounded by a TV crew. It was humid and hot and the noises of the nocturnal creatures outside were unfamiliar and disorienting. I was there to consume ayahuasca, a psychoactive beverage traditionally imbibed by Indigenous South Americans during spiritual ceremonies.

The TV crew was there to document the ceremony for my psychological YouTube show, *Mind Field*. Dr. Robin Carhart-Harris, then Imperial College London's Head of Psychedelic Research, was there to take measurements of my brain activity during the "trip." The whole event was exceedingly unusual—for both me and the attending shaman! But looking back today, the part of the experience I find myself revisiting the most is something that happened just before it all began.

Immediately prior to the ceremony, the shaman asked if I had a purpose. What was it that I wished to explore during the journey? A personal trauma, perhaps? A relationship? Understanding of the self? Escape from the self? I told the shaman that I hoped to visualize a fourth spatial dimension.

I was cautioned against that purpose. It was too impersonal. Too "thinky" and cerebral. Instead of a math problem, I was told, it was better to pick something transcendental. Wanting to "do it right," I chose a different purpose. But I often think now: What could have been more personal—more transcendent!—than escaping the meager three dimensions of space that I (and you!) are born into? What could be more humbling

and insightful than witnessing a direction beyond human perception?!

To this day, when I lie down to fall asleep, instead of counting sheep, I often attempt to imagine a fourth spatial dimension: an additional direction that isn't just some combination of up-down, left-right, or forward-backward. Lately, it takes the form of imagining not just one infinitely expansive, never-ending 3D universe, but two. The pair sit there, in my mind, as I try to mentally fit them "right next to each other," superimposed but not "touching" in any ordinary sense. So far, I have had absolutely no luck. Even worse, the nightly ritual hasn't yet (to my knowledge) affected my dreams. I reserve hope, though, that eventually my nightly meditation will result in a dream that gives me a vision, a glimpse, (oh! the briefest touch!) of a direction that isn't just up-down, left-right, or forward-backward. I want to look *ana-kata*. Those are the names Charles Howard Hinton (1853–1907) gave to directions perpendicular to the usual three we inhabit.

Toby Hendy and I share a fascination with Hinton's work. In 1888, he coined the term "tesseract" to refer to a cube of four dimensions. His descriptions of higher-dimensional shapes and how they might appear to us provided an early and still hardly bested attempt at allowing us to "see" beyond the three-dimensional walls we're each caged in. Like us, Hinton was eager to intuitively perceive higher spatial dimensions. His famous blocks ("Hinton's Cubes") promised to allow a user (after much practice and familiarization) to truly intuit and perhaps even SEE a tesseract. Eager for this reward, I collected his books and meditated on his exercises. Alas, unsurprisingly, I never achieved the visions I desired.

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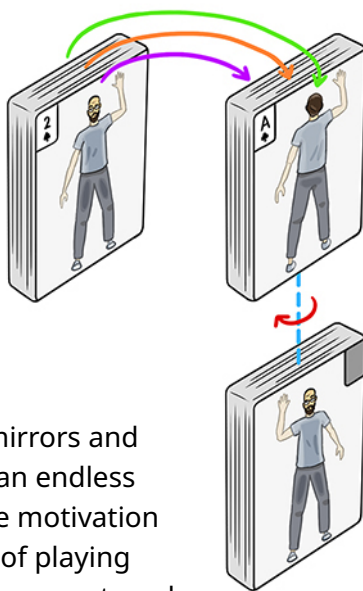
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I've heard that young children may have a better chance. Unlike us grownups, their relative newness to sensing and exploring space may mean they're not yet locked in to only ever thinking with three spatial directions. I find that plausible and exciting! But also, I don't care at all. Big whoop. I don't want an *infant* to directly behold a hypersphere. I want *myself* to!!

My obsession with hypershapes and ana-kata didn't initially come from an interest in seeing higher dimensions, though. It began with the bewildering world of *chirality*. I read Martin Gardner's *The New Ambidextrous Universe* and was fascinated by what exactly makes some mirror-images superposable but not others. In a video I made many years ago, I asked, "Where are your fingers?" Today, however, I wonder less about where they are and more about *how they are different*. My right hand and my left hand are obviously not exactly congruent: their creases and freckles are unique. But their general shapes are tantalizingly similar ... BUT NOT QUITE. If I try to lie one on top of the other so that they match, I have to either turn one over so that its palm faces the other way, or I have to point the thumbs or the fingers in different directions. It's ridiculous. Now, surely, the difference between my right and left hand isn't actually that mysterious: They just happen to have their thumbs *on different sides*. Yes, but now consider the problem in two dimensions.

The letter "L" can be rotated 180° around a point in its plane to become a "7", but it cannot be rotated around a point in its plane to become its own mirror image: "J." Try it yourself! If, however, we are allowed to lift the L from its plane while rotating it through a

third dimension around *a line*, we can easily turn L into its mirror-image, J. Similarly, if my right hand could be rotated around *a plane* through a fourth dimension, it could be easily turned into its mirror image: the shape of my left hand!



The connection between mirrors and higher dimensions has been an endless fascination of mine. It was the motivation behind *The Body Deck*, a pack of playing cards I invented. On each transparent card, a coronal slice of a human body is printed. Stacked together in the right order, a 3D human form appears to be standing 'inside' the deck raising its right hand. But reverse the order of the deck and the person is suddenly looking the other way and is raising their left hand! By reversing the cards, you have effectively turned the person "inside-out." You have swapped their front and back sides without changing the positions of their left, right, top, or bottom. This maneuver can easily be done if you have access to a fourth spatial dimension. But in our 3D world, it can only be done destructively and partially: by literally picking an object apart and rearranging its pieces. The completeness of the transformation will be limited by how small you're able to make the pieces.

We miss out on a lot by being stuck in 3D. But, if you want to feel better, remember that your everyday life would be unimaginable to a 2D creature. You live in a

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world they can't even DREAM of! I can imagine infinite 2D surfaces coexisting right next to each other. That's just a stack of papers. But to a 2D creature confined to 2D movement, the idea of a *stack* of surfaces where some are *above* and others are *below* would be completely incomprehensible, just like the stacked universes I try to imagine when falling asleep are to me! To a 2D creature, even the lowliest, most boring parcel of 3D space is a mystery so unreachable that for them to conceive of it would be – as Toby explores in this very book – nothing short of a miraculous and religious experience!

Indeed, the work of mathematicians in the late 19th century was of special interest to spiritualists desperate to find a way to locate Heaven and Hell and angels and demons. If spirits and their otherworlds were not literally above us or below us, but were instead *ana* and *kata*, that would solve everything! By situating ghosts and the supernatural outside our three dimensions and, instead, *adjacent* to them, the spiritual world could be both *nearer* and yet just as *unreachable* as they seemed to be.

Today, we “look” in these directions for modern-day gods: “cracks” through which gravity may be leaking. “Shortcuts” between otherwise distant locations in space. Perhaps each of our lives is just a 4D “worm.” A tight stack of sequential 3D “slices” strung *ana* and *kata*, like pearls on a necklace. Under this conception, four spatial dimensions exist and one of them is what we call “time.” The world as it was in 1986, at the very moment of my birth, is part of this 4D object. It isn't above or below me today, it is *kata*. *Ana* from that moment of my birth lies what *you* currently call “now”: everything that inhabits the moment in which you are currently reading this

sentence. We have no way of seeing the whole 4D shape at once, for our awareness is somehow stuck in just one 3D slice at a time. Like characters in a flip book, as time marches on we inexorably move from one slice to the next. The whole show produces in our minds the illusion of motion and of a passage of time. Of then and now. Of velocity. And yet, in reality, there is only *the shape*, the 4D hyperbeast we call “history”, whose cross-sections we call “moments.”

I spent months trying to work out (and to this day occasionally revisit) how the physical laws we learn in school would appear if successive moments of time were arranged, one on top of the other, in a 4D “stack.” An elastic collision, a pendulum, a falling apple. Thought of in this way, conservation laws become wonderfully symmetric shapes. Do they reveal anything about mass? About momentum? As of now, I have nothing to share. No insights and no revelations.

I felt better after talking to Toby a couple years ago at an event for educational creators. For a long time, I had been ruminating privately on dimensions. I’d read every book I could find; I’d filled journals with sketches of 2D creatures, their experiences with 3D objects, and what that might mean for me and 4D things. It was fun ... but lonely. Just before the convention, however, Toby started making some incredibly popular videos about life in a 2D world. Finally! Someone I knew I could talk to about all this stuff! When we met up, we shared ideas and swapped book recommendations. The works of Rudy Rucker played a starring role in our conversations. We agreed that his books contained some of the clearest (and most fun) illustrations of higher dimensions we’d

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ever seen. Most importantly, Toby and I were able to commiserate together. Like me, she also yearned to actually SEE beyond 3D. And, unlike so many others, she hadn't given up! She shared with me a conviction that it *might* somehow be possible. When I found out that she was writing a book (this book!) about exploring dimensions and she asked me to write this foreword, I was absolutely thrilled. If anyone ever figures out a way to mentally picture higher dimensions, it will be because of books like this one in which life in other dimensions is depicted accurately and concretely. Toby has included in these pages more examples of trans-dimensional analogies than I've ever seen anywhere at once. How would a shadow-puppet show work in two dimensions? What about baseball in four? By considering these translations, we exercise the space parts of our brains and grow stronger at intuiting the natures of that which lies beyond us.

In the pages that follow (and in the videos she makes), Toby plays with mathematical ideas like pieces of playground equipment. At the end of the day, that's ultimately what they are. Math is often taught to us early on as if it's a puzzle that's already been solved. A dance whose steps we must remember and rehearse. But like any dance, you can also just move your body in the way you want. In another, equally true way, math is a collection of monsters and we're all little Dr. Frankensteins. We discover math as a timeless logical blob, but then carve it up into shapes of our choosing. We play with them, and then watch what happens. These shapes (and the rules we set for them) can be, as physicist Eugene Wigner famously

observed, “unreasonably effective” at explaining and predicting the world we live in. But sometimes they’re unreasonably effective at describing and suggesting worlds we don’t inhabit!

For that reason, I love that the title of this book works in two ways: *A Guide to Making Friends in the Fourth Dimension* can mean becoming friends with 4D creatures, but it can also mean making friends in the sense of *creating* them—conjuring them up from scratch and then giving them a hug.

With this spirit, mathematics becomes re-creational and we, the players, become god-like. The history of math and science is a story of taming the world, but it’s also a story of finding new ones. As we search and question, we encounter more and more things that lie beyond our perceptual abilities: X-rays, hyperspheres, quarks, infinite sums. When you sit back and consider all the things we can manipulate but cannot hold (neutrinos, genotypes, momentum) and all the things we can envision and describe that may have no real existence (20-dimensional cylinders), it’s staggering just how little of space and time and matter we really do perceive. We aren’t the universe experiencing itself. If anything, we seem to be the universe *ignoring* itself. We are the universe looking for something else.

Michael Stevens
Los Angeles, 2025

INTRODUCTION

Thank you for picking up this book. On the pages before you lies an exploration of recreational mathematics. *Recreational* and *mathematics* might seem like two words that don't belong together, but that's because math is usually dressed up in its most serious tax-accounting outfit when we meet it. However, math does have a fun side and can be something more than a tool.

Together, we will travel across spatial dimensions, meeting all kinds of strange characters along the way. Although these characters are fictional, our interactions with them will shed light on real mathematical ideas.

In the pages ahead, we will think about the meaning of the word *dimension* and then stop to ask ourselves, "If a two-dimensional world were to exist, what would it look like?" After that, we will explore the more familiar world of three dimensions before traveling to a hypothetical four-dimensional space. The last part of this book ponders what might lie beyond the fourth dimension.

The goal of this book is to leave you thinking about complex ideas in a way that you might not have considered before and to show you that mathematical concepts can be fun and creative. We experience life in three spatial dimensions, and while you will see that higher and lower dimensions can be mathematically described, their properties are mostly speculative. With this in mind, I ask you not to take any passages from this book too literally.

Two-dimensional creatures don't exist in our world, but imagining how their world might work stretches our mental muscles and problem-solving ability. This kind of thinking could even have a profound effect on the course of your life.

Numerous scientists, including Stephen Hawking, Lisa Randall, and Roger Penrose, built their careers on exploring dimensions. Claude Shannon, often called the father of information theory, introduced the concept of encoding data within multi-dimensional “spaces” of signals and noise. Countless others have found applications for higher-dimensional concepts in fields as diverse as computer vision, materials science, quantum computing, game design, and architecture.

If you are hoping to avoid calculations, then you’re in luck: This book will not contain many equations. In Part 4, there is a chapter titled “Mathematics of the Fourth Dimension,” which contains a few. You can choose to skip it if you like. I have used the word *mathematics* a few times already to establish what this book is about, but equations and proofs are only one part of what mathematics is. Imagining shapes, constructing analogies, and thinking logically play much bigger roles.



I have one little secret that I wish to share with you: My relationship with mathematics has not always been passionate. At school, I was more drawn toward subjects like English and the arts, where creativity takes center stage. But math had this intriguing way of pushing my limits.

There is a belief I hold dear: The finest things in life often emerge from embracing what’s difficult. In pursuit of this, I majored in physics and mathematics during my university years. I have continued to be pushed and intrigued by mathematical ideas ever since.

The stuff we learn at school is just the tip of the mathematical iceberg. There’s a whole world of big ideas

in math that go beyond the numbers and equations we're used to. I'm not claiming to be a human calculator. (Far from it: My basic math skills are pretty terrible.) Don't ask me to calculate change at a restaurant, but if you ask me to sit down and ponder higher dimensions with you, I could write a book. (And I did!)

When I was a teenager, TV documentaries and YouTube science communicators kindled my curiosity for these subjects. I am grateful to have come full circle, having carved a path for myself sharing math and science ideas on YouTube and TikTok. I know how powerful accessible online content can be when it comes to planting the seeds of academic curiosity.

I have found my niche sharing mathematics on my YouTube channel, Tibees. Some of my online videos take a leaf from the book of legendary painting instructor Bob Ross and establish a calming learning environment where mistakes are seen as "happy accidents" rather than something deserving punishment. In the process, I created a series of videos about higher dimensions, and, after encountering boundless curiosity and questions from you, the audience, I wrote this book in an attempt to answer as many of those questions as possible and provide a source for further exploration.

Of course, I am not the first to ponder these questions. Like many before me, I was inspired to think deeply about visualizing other dimensions by Edwin Abbott's classic book, *Flatland*. *Flatland* was published in 1884 and follows the life of A Square living in a two-dimensional world. My own depiction of lower-dimensional worlds differs from Abbott's vision but follows similar thinking to explore higher dimensions by way of analogy.

The fiction and nonfiction works of American mathematician and author Rudy Rucker have also provided inspiration in

the shaping of this book. *The Fourth Dimension: Toward a Geometry of Higher Reality* (1984) first taught me how to draw a Tesseract, the four-dimensional analog of the cube.

This book builds on the ideas of those who came before it but spins new fictional tales and introduces ways of thinking about and playing with geometry, physics, and other mathematical ideas. Some of these embellishments were inspired by my viewers on YouTube and TikTok.

Both of the authors mentioned above managed to write about math in a way that captured my imagination and inspired me to pursue it further. In the right light, mathematics can be so much more than just numbers and formulas. It can be recreational, fun, meditative, and a fantastical escape from reality. I hope that this book offers you just that.





PART 1:

ALL ABOARD!

The *Vessel of Higher Dimensions* exists not in physical space but in your mind. Rather than offering concrete directions on how to get to the fourth spatial **dimension** by following an earthly map, I must ask you to imagine your way there. I will guide you, and, together, we will chart a course through the unfathomable depths of geometry.



This course may seem detached from the practicalities of our three-dimensional lives. You might wonder, how will understanding dimensions help me pay my bills?

Well, I'd like to offer a few reasons why you should join me on this tour. The fourth dimension, and those beyond, have seeped into our cultural consciousness. The fourth dimension is mentioned by mystics and those attempting to bridge the gap between this world and the next. Higher dimensions are outside of our sight, and if they did exist, it is unlikely we would ever be able to interact with them. This makes them fertile ground for untestable claims.

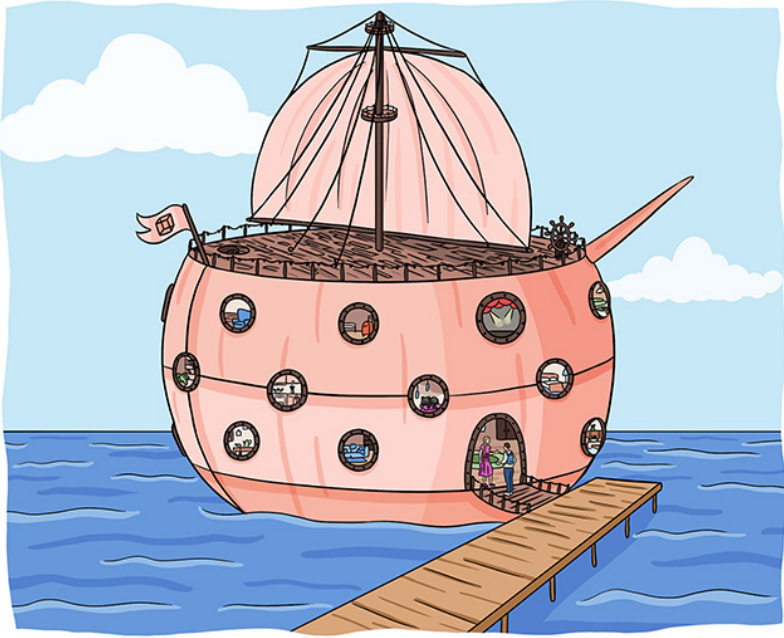
Often, these mentions carry a mist of confusion, clouding the real nature of the fourth dimension as a mathematical concept. This book will equip you with enough clarity to see through that mist, to understand and appreciate the abstract beauty of the fourth dimension for what it is.

I will introduce you to my weird and wonderful friends living in all kinds of strange places. You will see a world of structured whimsy, built on logical laws and patterns.

We will be diving into a fantasy novel that isn't confined by an author's imagination. Mathematics is an open-world adventure where you can carve out your own narrative, a story that's not only reproducible but open to your unique insights. Mathematics is a form of escapism, yes, but one that enriches the mind and tickles the soul.

My name is Toby, and I will be your tour guide today. Our ship is poised at the harbor of understanding. So, dear friend, it's time to hoist the anchor and set sail. A word of warning, though: Adventure, like a pot of water left on the stove, has a tendency to boil over when you least expect it. Be prepared.

LEAVING THE SHORE OF 3D



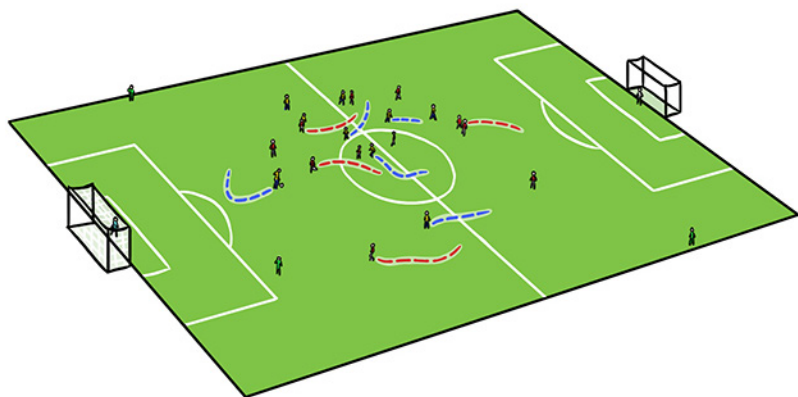
There is our vessel, that bulbous pink **sphere**, waiting at the end of the jetty. It's bigger than it looks; there is plenty of room inside for all of us, and even room for my cat, Lemon. Follow me inside. There is a spiral staircase from top to bottom and portal windows at every level. There's Lemon, sleeping in her window hammock. I advise positioning yourself at any other window—you know how cats can be with strangers. I do hope we will all be friends before long.

As we push away from the beach, it is prudent to check our compass and confirm that we understand our navigational landmarks. The word *dimension* can describe just about anything, such as time, speed, and mass. However, in this book, *dimension* refers to space and our level of freedom to move within it. I'm not referring to any political

notion of freedom but rather the freedom to move forward and backward, left and right, up and down.

Those are our familiar three dimensions. For many of us, unless we're floating in a hot-air balloon or scuba diving, our everyday experiences can seem largely confined to two dimensions. Despite being creatures of a **3D** world, we often move like characters on a flat stage, especially when our actions are observed from a distance.

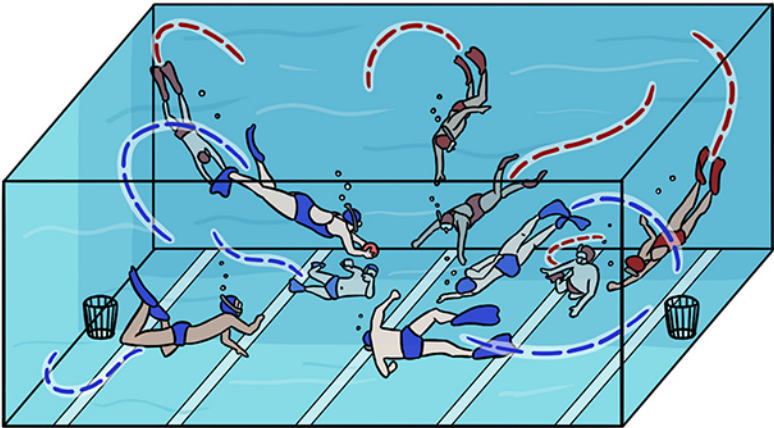
Take, for instance, a football match viewed from the lofty heights of a stadium. The players might seem like miniature figurines on the green expanse, zipping around mostly on a horizontal plane. Every so often, the ball is lofted into the air, introducing a brief taste of the third dimension. A player may even leap to execute a header, but such movements upward demand more effort and cover a much smaller range compared to running left, right, forward, and back.



One game that makes good use of all three dimensions is underwater rugby, a unique sport I've experienced first-hand. Picture this: players gliding through a five-meter-deep pool, cradling a weighted ball. Each team has a goal at the end of the pool, and as players swim towards it, an opponent

can swoop in from any direction. You could be tackled from above, below, or any side and have the ball stolen from you.

It's a contrast to sports like water polo, where despite the same aquatic setting, the game is largely played on the surface. In water polo, you don't have to worry about someone sneaking up from below you. Underwater rugby is a game that truly embraces the idea of three-dimensional freedom.



Some of you onboard might call time the fourth dimension, but this is not a time-traveling ship. Picture time as our invisible and constant companion, moving always forward, pushing us unceremoniously into the future.

Yes, in the language of physics, it is commonplace to refer to time as the fourth dimension. This concept of **spacetime**, a model combining three dimensions of space and one of time, was formalized by Hermann Minkowski and built upon Albert Einstein's work on special relativity. But if there are four cups of tea in front of you, then which is the fourth? Well, it depends on where you start counting. Time could just as easily be called the "zeroth" or "first" dimension

without anything changing. When we talk about extra spatial dimensions, we are adding more cups of tea to the table.

Let's cast our minds to a world of two dimensions. You could imagine cartoon characters confined to live on a screen. Even in that flat existence, time continues its march. There, time represents the third dimension, still ticking along as the residents navigate their **2D** world.

Think of it this way: Time is always the plus-one to the party of spatial dimensions. It doesn't matter how many spatial dimensions are present—two, three, four, or more—time will always tag along as an extra. As we dare to venture into the fourth dimension, we'll find ourselves in a landscape with four spatial dimensions, with time as the fifth companion.

To clear any potential clouds of confusion on this voyage, let me be unambiguous: Whenever I refer to the fourth dimension, I am referencing a fourth dimension of space. I do encourage you to scrutinize the context if you ever find yourself on other mathematical tours.

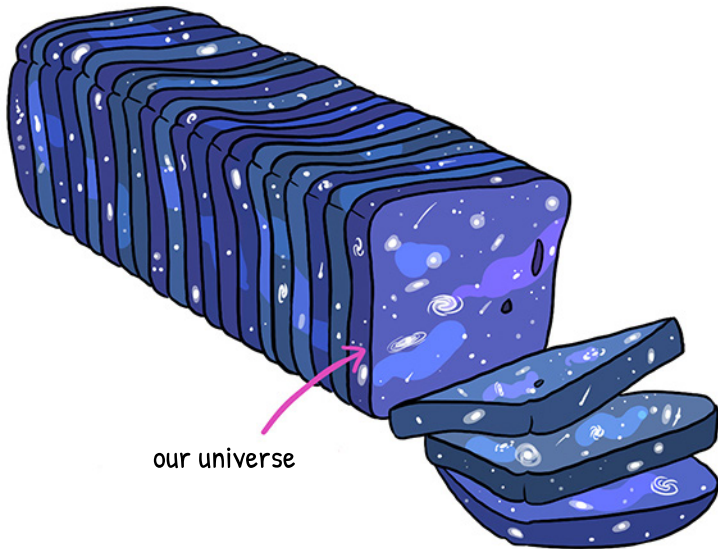
Hanging above the door here is our ship's motto:

One can never have enough socks.

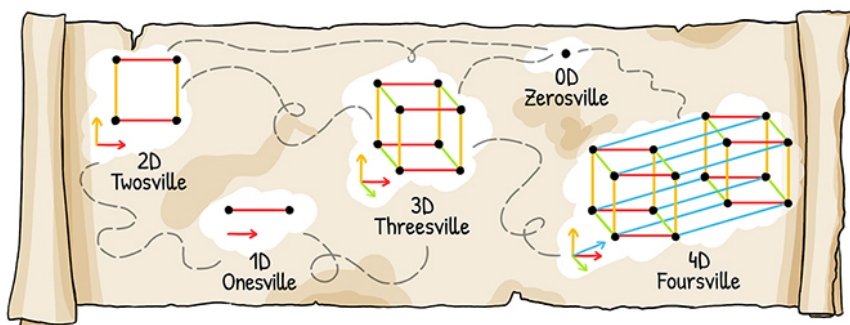
I hope yours didn't get too wet crossing the beach this morning. If they did, then you can borrow some of my spares; I have a whole pile. When traveling between dimensions, things can often go missing, and socks are one of the favorite targets of four-dimensional thieves.

There is a spot up on our round roof where you can wave goodbye to those who are not joining us on our tour today. As we leave, I feel melancholy, considering the limitations in our current space of existence. If any friends came to wave you off, consider that you are leaving them trapped in a world of mere length, width, and height.

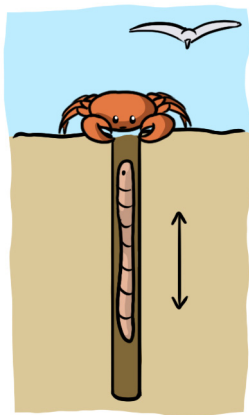
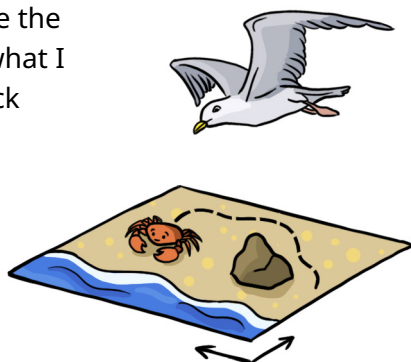
In this existence, everything you know and love is but a slice, a mere **cross-section** of a much grander whole. It's as though we're living within a slice of bread from a vast cosmic loaf. Our slice is comfortable and familiar, but not the entire loaf of the universe. Our perception of the world around us is sadly limited. Our brains, such amazing organs, can only paint a picture within the constraints of 3D space, even as we suspect the universe may contain unseen dimensions, curled and hidden like shy little creatures in the forest of reality.



We can move forward, backward, left, right, and up and down. To us now, the idea of moving through a fourth spatial dimension seems as ludicrous as wearing a platypus as a hat. I have mapped out a path for us today that I believe will give us the best shot at imagining a world beyond 3D. It may be counterintuitive, but in order to reach that higher place by this evening, we first want to start on a little empathy tour, paying a visit to those who have much less than us.

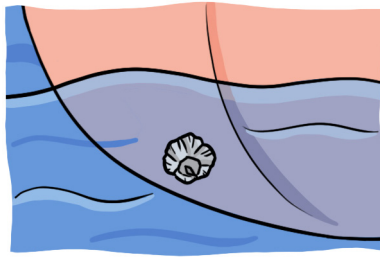


We don't even need to leave the beach to get a rough idea of what I mean. From the roof, gaze back toward the sand. Imagine a humble critter on the beach that is bound to the flat plane of the sand and can move in two directions: forward and backward and left and right. If it encounters a rock, it must find a way around, rather than over or under. This creature, confined in its movements, might easily fall prey to a bird it never saw coming because it lacks vision in the third dimension—that of height.



In a similar vein, ponder a one-dimensional worm, a living line burrowing through the sand, only able to move up or down. It cannot approach the sea, nor can it venture inland. Its world is one of vertical ascents and descents, a life possibly shortened by a critter whose approach it sees but cannot avoid. Its glimpse of the critter is just a worryingly dark dot.

A barnacle growing on a rock beneath us experiences life in zero dimensions. It is fixed to the same point at which it first attached. It cannot move forward, back, left, right, or up or down, and it sees nothing. It lives its entire existence stationary, relying on sustenance to drift towards it in the current. That is unless the barnacle is attached to the hull of our ship: then, it would be bound for a mind-bending voyage.



Luckily for us, we are on that journey now. I have pulled the last rope in from the jetty and we are starting to drift. Find a good seat and steel yourself for the voyage. I hope you're wearing your most comfortable and awesome socks, dear friend. Things are about to get quite strange.

THE LINE

After years burrowing on an incline,
I was about to become divine.
Peeking through the dirt came sunshine,
That would crown me The Chosen Line.

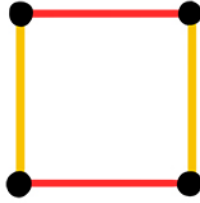
But I surfaced and yelled a prayer,
As I was face to face with a square,
A shape that had three lines to spare,
My envy was more than I could bear.

Its approach I could not foresee,
And with nowhere to flee,
I let it consume me.
My lineage reduced to something tasty.

The square glided across a plane,
In its dark belly, I felt disdain.
I'd thought I was the top of the food chain,
But I was just a speck in this domain.

We stop. Is that screaming I hear?
Something strange has eaten the square!
In a motion to which nothing can compare,
We are flying, in a cube, through the air.

The movement made me spew,
I saw a point that I'd forgotten to chew.
And realized there is always someone better,
And someone worse off than you.



PART 2:

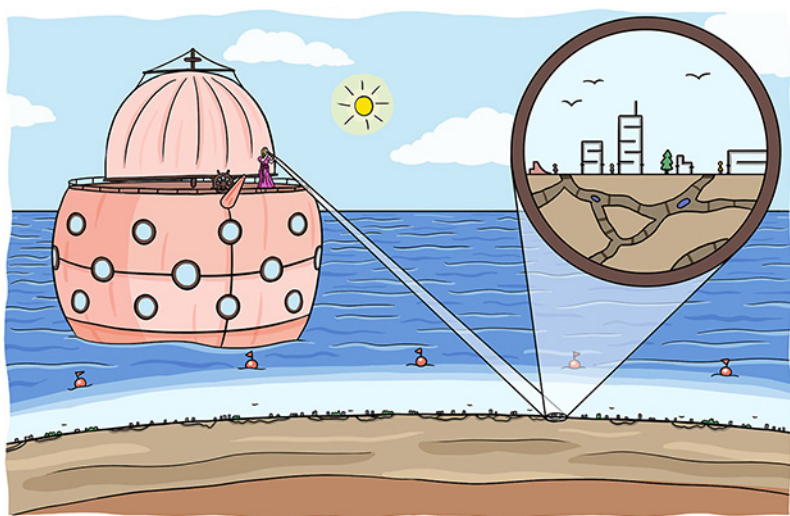
TWOSVILLE

Our first stop is a land confined to only two **dimensions**. Its location is not one we could pinpoint on a map. To get there, imagine us sailing now, moving away from the comfort of the shore.

The ocean's waves grow quiet until the water is completely still. Now, our ship is floating on a smooth mirror of water, stretching out endlessly in all directions. If you glance down, you can even spot our reflection on the surface. Above us, there's the wide, open sky, and beneath, a whole underwater world teeming with life.

Now, take this special spyglass I'm handing you. It's a telescope that helps you see light that is usually confined to another dimension. It can see the thinnest rays of light coming from the smallest of things. Use it to look at the water's surface again. This time, you'll see something astonishing. We're not just floating on water; we're floating on a world made of lines and shapes.

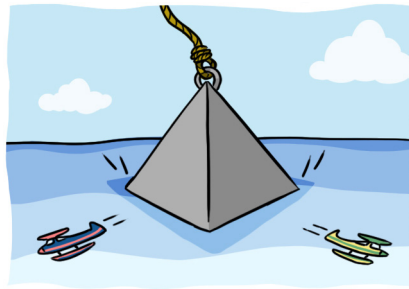
Welcome to a world as flat as your least favorite pancake but undeniably more interesting. This is **Twosville**, a world



where existence is squeezed into the thinnest slice of space, floating on the surface of the flat ocean.

We often take our three dimensions for granted. It's a shared failing to not fully appreciate what we have until we consider those who have less. Similarly, it is difficult, if not impossible, to imagine what we've never seen or experienced. My favorite part about bringing guests to Twosville is that they always leave with a new appreciation of all three of our dimensions back home.

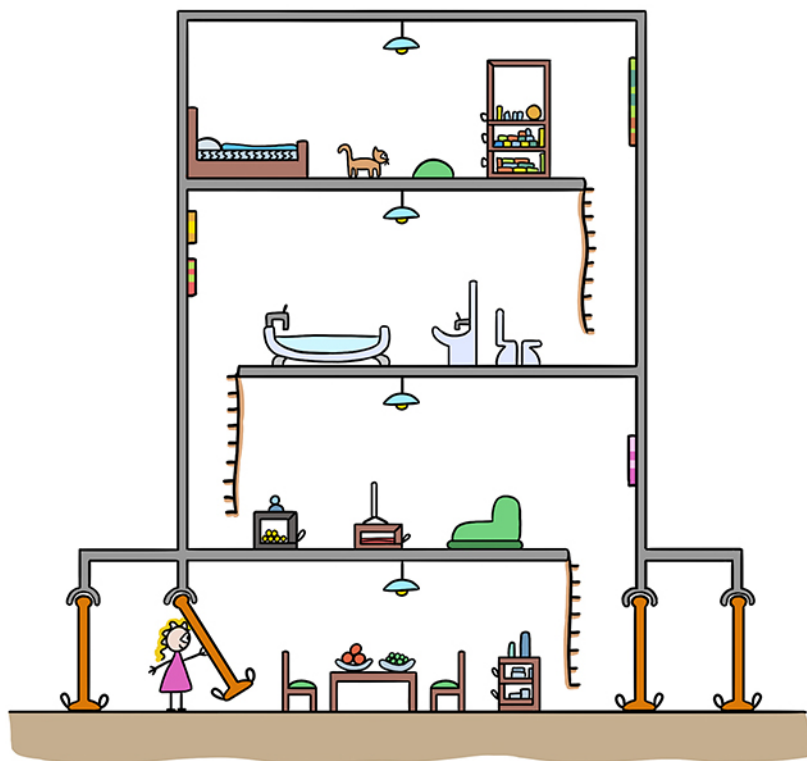
Please join me in gazing down upon this new world. I carefully prepare to drop our anchor, a hefty object shaped like a triangular pyramid. But before casting it into the depths below, I pause to ensure the waters are clear of any two-dimensional aircraft. In the past, less attentive captains have recklessly thrown their anchors overboard, only to unwittingly collide with these flat vessels.



Such incidents have been alarmingly frequent, leading the residents of Twosville, whom we affectionately call **Flatlings**, to dub this area of danger the Bemusing Triangle. From the Flatlings' perspective, aircraft mysteriously disappear without a trace. Little do they know, the culprit is often the triangular **cross-section** of our three-dimensional anchor intersecting their plane—a fleeting but devastating encounter in their flat expanse.

LIFE IN TWO DIMENSIONS

Direct your gaze to that nearby house, a simple rectangular design. It has two double-door entrances, one on the left and one on the right. The inhabitants of this fascinating 2D world must exercise extreme caution to securely fasten the door behind them before moving through the next.



This action is no mere convention of manners or practice of energy efficiency but rather an absolute necessity of architectural integrity. Should both doors on one side of the house be left ajar simultaneously, the house would fall. Such an event is a rarity, but I recall an incident where I bore witness to just such a calamity.

A certain parent, perhaps a tad more relaxed or maybe a touch more forgetful than most, had overlooked installing appropriate child locks on their doors. During the busyness of the day, a curious toddler managed to open one of the doors. At the same time, the parent was engrossed in receiving a hefty stack of mail at the front. The result, though not catastrophic, was a sight to behold. The house came to rest on a sharp lean, with furniture and belongings thrown to one side.

But fear not, for disaster was averted, with no injuries to report. A crane operator, a hero in this 2D world, was summoned swiftly to lift the sagging top and hold it up. At the same time, the doors were repositioned. In a 2D world, tasks we perform thoughtlessly in our 3D world become intricate dances of balance.

Flatlings are remarkably nimble, perhaps due to their light weight. The topology of their world necessitates a constant ballet of leaping and climbing to navigate through the cityscape. I daresay they have, in this aspect, usurped even the best of our 3D parkour champions.

A casual stroll here is tantamount to an urban obstacle course. Flatlings can vault over an entire dwelling with the grace of an Olympic high jumper. But such an action, though physically possible, is deemed uncouth. It's the 2D equivalent of stepping over a picnic in progress—quite an affront to their cultural sensitivities.

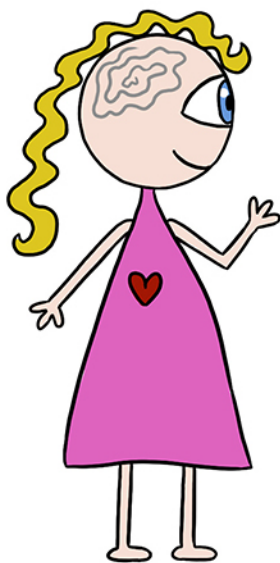
The polite thing to do, according to the customs of this peculiar society, is to simply knock on the door and pass through the house. As guests in this 2D world, we may find this strange. Surely bounding over a rooftop is less intrusive than traipsing through a stranger's living room? Yet we must respect their ways.

Take a moment to study the unique physical forms of the Flatlings. They're like the simplest stick figures you can draw,

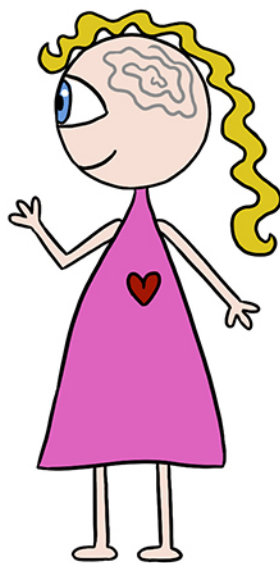
with triangular dresses or boxy shirts attached to a circular head, garnished with two arms and two legs.

The arrangement of their eyes is particularly fascinating. The occupants dwelling on the left side of this flat town bear an eye on the right side of their face, giving them a forward-facing view into the center of town. On the other hand, those residing on the right of the center line have an eye on the left side of their face—a mirror image of their left-side counterparts.

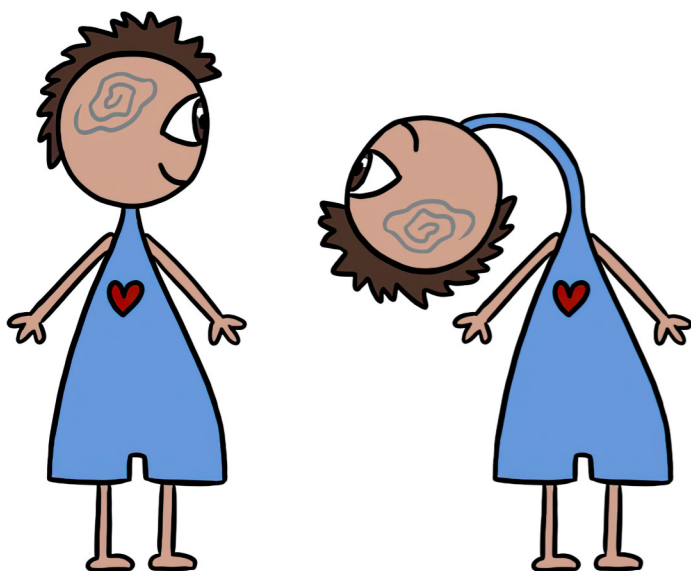
Left-sider



Right-sider



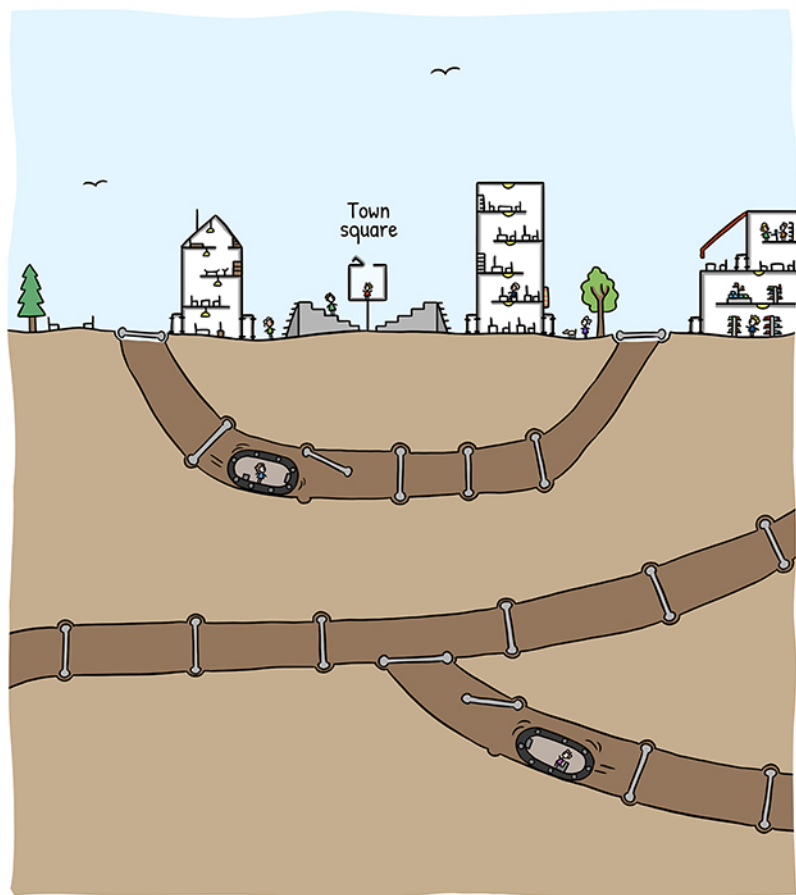
It is important for Flatlings from both sides to regularly train and stretch their neck muscles for the occasionally essential task of looking behind themselves. This involves tilting their heads all the way back until their upside-down face points in the other direction.



In the center of town is the town square, which serves as a meeting place for citizens from both sides of town. It's an intersection of perspectives where many debates between left-siders and right-siders take place.

However, outside this communal space, inhabitants prefer to remain within their familiar territories. It prevents too much reverse movement—not that they are incapable of it. In fact, they are surprisingly adept at backward parkour (thanks again to their nimbleness).

If you are wondering why the town square is so busy, it's because it serves as more than just a meeting place. It is also a place of deep cultural significance. The Flatlings spin tales of a strange phenomenon occurring in their town square. It is rumored that, on rare occasions, standing precisely at its center may inexplicably flip an individual from a right-sider to a left-sider, or vice versa.

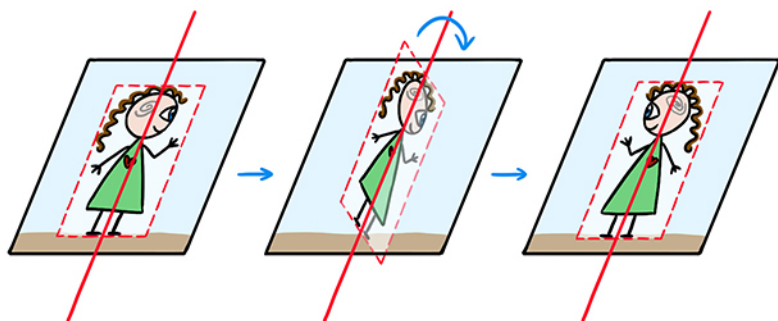


Children often playfully take turns standing at the square's heart, posing for snapshots before leaving unflipped. Those who claim to have witnessed the last alleged flipping tell tales of a petrifying spectacle. The individual at the event's epicenter seemed to vanish entirely, except for a thin vertical slice of their body, before reappearing intact yet inverted.

Onlookers claimed they were able to deduce the body had become a thin sliver because it took on a translucent glow, allowing light to pass through it from the other side for a brief

moment before returning to normal opacity. Some reported seeing a small red dash in the center of the person's chest that looked suspiciously like their heart, fully exposed at that instant.

From our superior vantage point, this myth becomes tantalizingly plausible. If we were to rotate a segment of 2D space through the third dimension, the section being flipped would indeed appear to thin into near-invisibility, except for a narrow axis along which it rotates.



Look below the bustling heart of town, and you will notice a curious network of subterranean tunnels. These are crucial transport arteries enabling fast movement across town. These underpasses are lined with a series of supports, placed with utmost precision every 10 meters. Each support acts as a gate, enabling passage from one end of the tunnel to another. Before proceeding further, one must open and close each gate in sequence, essentially unlocking and locking a series of doors. The weight of the world above rests securely on these gates, which, when shut, maintain the integrity of these tunnels.

The need for such a contraption might baffle us, as we are used to tunnels in our own 3D world that can be carved through rock, without the need for such gates. But here's where we must remember the extra dimension we enjoy, where the tension in the rock from that additional spatial

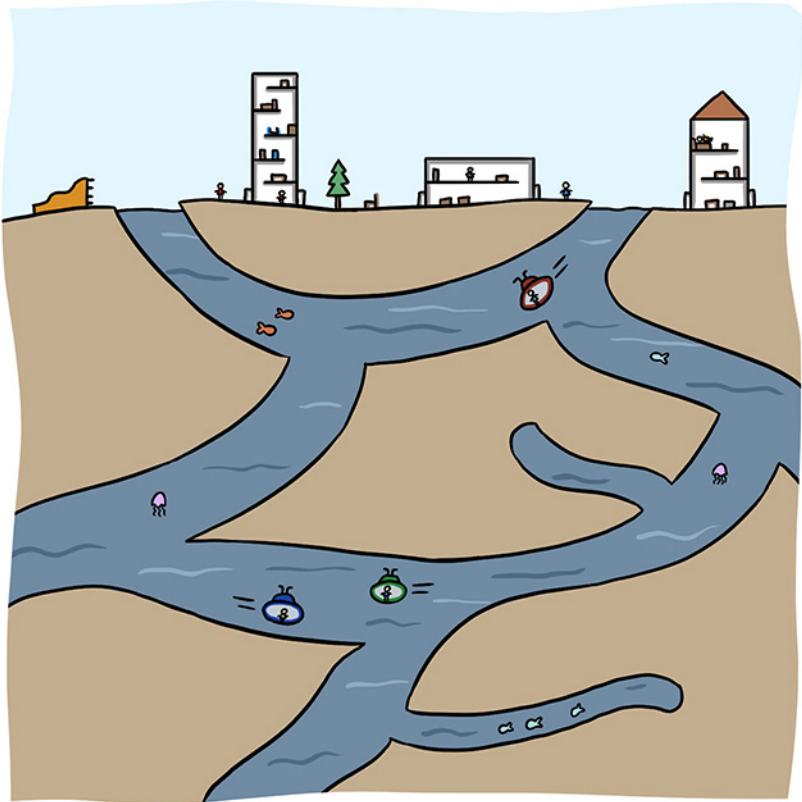
dimension holds our tunnels firm. In this two-dimensional world, however, such concepts are absent. The gates, thus, are not merely a feature but a necessity.

While the neighborhood nearest the center of town relies on these road tunnels, older neighborhoods farther out from the center have historically used a different method. As we can see from looking down from above, certain areas are built upon a naturally occurring underground cave network, most of which is flooded with water. Submarines were once the most popular form of long-distance transportation in Twosville, diving through these underground caves. The water keeps the land above stable without the use of gates, and submarine traffic can even pass in different directions without issue.

These older parts of town have a romantic pace to them, much like Venice. In recent times though, the inner city has become dominated by the honking, tooting, and gate-slamming of cars.

As intriguing as their architecture is the anatomy of the Flatlings themselves. As you might imagine, existing in this flat plane brings some rather unique physiological challenges. In our three-dimensional world, we take for granted a digestive tract that runs right through us, a handy biological feature that allows for easy and efficient nutrient processing. But in a two-dimensional world, a hole that goes all the way through an object from top to bottom would separate it into two pieces.

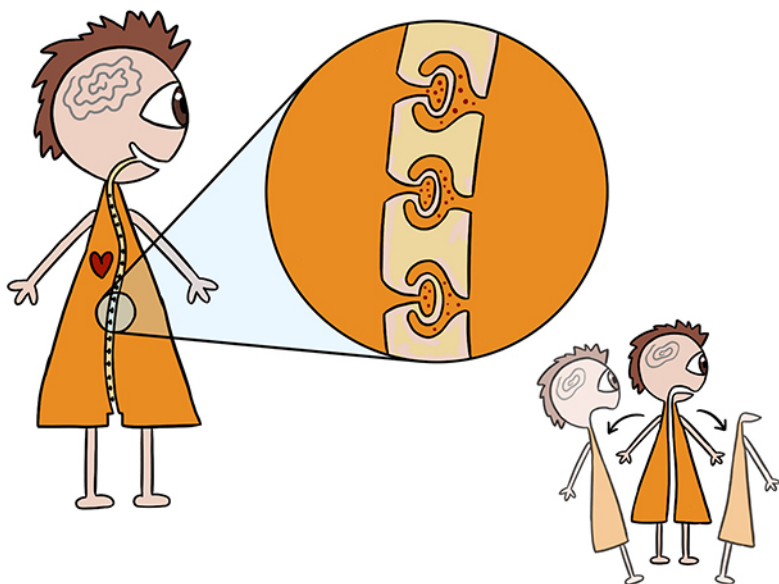
Think about the digestive tract as a pair of scissors, and the Flatling as a piece of paper. If the scissors make a snip at the top of the paper and a snip at the bottom, can you think of any way that the scissors can cut through from one snip to the other without causing a chunk of the paper to become disconnected? I don't think you would be able to.



A digestive track like ours would similarly split a Flatling into two halves. There would be no lines joining the two halves together, and so the disastrous outcome would be that the 2D creature would fall apart like a cut sheet of paper.

To adapt to their unique environment, Flatlings have evolved a self-gripping intestinal system. It is a tightly interlocked structure where the left and right halves of the creature hold each other together, preserving the integrity of their forms. As food traverses this system, each intestinal “gate” operates in a meticulously timed sequence—opening and closing in isolation—ensuring the two halves of the creature stay connected. The semi-permeable and delicate

folding of the intestine also allows blood and nerve signals to pass between the two halves of the body.



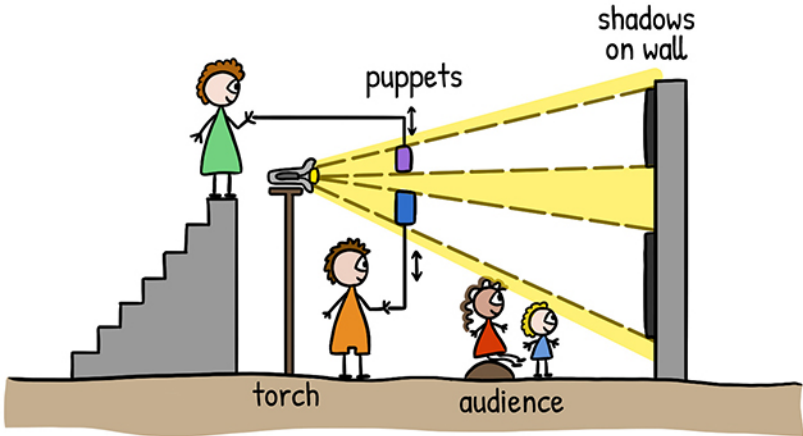
Some animals also exist here. There are flatworms and jellyfish, and even a creature that resembles a cat. Some of these creatures employ a more simple digestive solution: just one orifice used for both entrance and exit materials.



Be careful not to underestimate the intelligence of Flatlings. Their brain is made of neurons that are unable to cross over each other but that form complex networks nonetheless. Our own brains are somewhat sheet-like in their formation, folded intricately through the third dimension. The lines

of neurons in the Flatling brain are, likewise, so intricately folded that the details appear **fractal**-like.

I'd like to show you one of my favorite venues in Twosville next. It's a shadow-puppet show. In the 2D theater, Flatlings sit atop stacks of pillows, their gaze fixed intently on the flat, vertical wall that serves as the screen. A few feet behind the audience, a puppet master assumes their position. They stand just in front of a torch mounted on a pole, holding a stick puppet that, when moved into the torch's light, casts a flat vertical line of shadow on the wall.



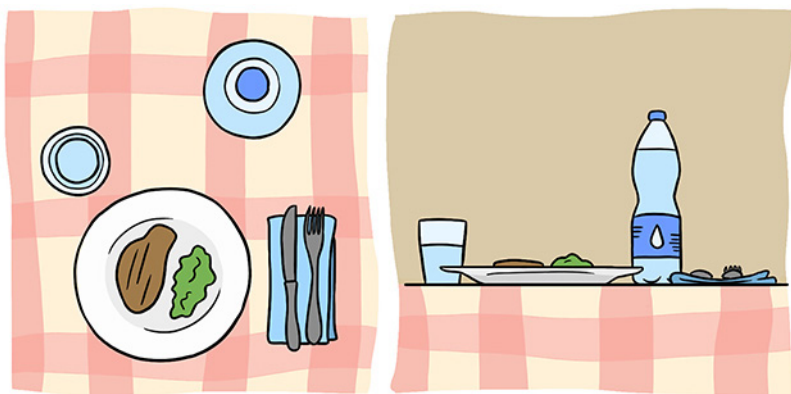
Standing even further back behind the torch is another puppet master with a second puppet on a long stick. This second puppet master extends their puppet over the torch to intercept the beam of light from the other side. The two puppet masters together are able to project two lines of shadow that move up and down the wall, talking to each other and playing games. To us, it might seem so simple to be devoid of entertainment, but to our two-dimensional friends, watching these one-dimensional shadow lines interact is very enjoyable.

In the current show, the two lines are arguing over who deserves to be Queen of Lineland. Moving up and down, they crash into each other in an attempt to gain the highest position in the line's monarchy.

FLATLING VISION

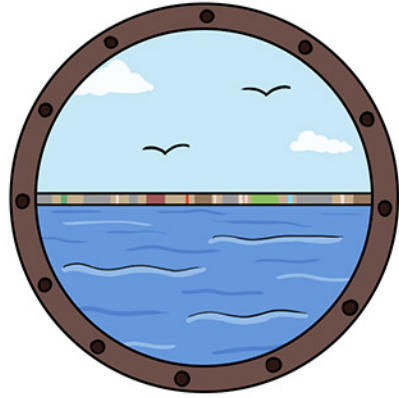
Our current bird's-eye perspective of the Flatlings differs from their own view of each other. To fully experience Twosville for ourselves, we must "descend" from our three-dimensional viewpoint and meet the 2D creatures at their level.

If you think of the 2D world as being laid out on the surface of a dining table, then currently we are looking down upon the table much like a dinner guest would look down at their plate. We will soon shift to have the perspective of a small child who stands beside the table with their eyes level with its surface. They won't see the full meal laid out, just the white sliver of the side of the plate, and the silver slivers of knives and forks.



Follow me down the stairs to the portal windows in the middle of our vessel. Our furry friend, Lemon, has sought

a different hiding spot, leaving her coveted viewing location available just for you. From this window, which rests exactly on the plane of Twosville, we will be like the child with eyes at table height. Unlike the view from our vessel's rooftop, where we looked down on Twosville, here at the central portal



window we now look out with the eyes of a Flatling. Our vision of this world is now as thin as a page from that unread novel that's been gathering dust on your shelf.

Our perspective here on the plane is dramatically different. Once used to the depth and variety of a wide field of vision, we are now restricted to a **linear** view. We can only see the world from the side, each object a mere line segment. It is similar to the shadow-puppet show that we witnessed earlier.

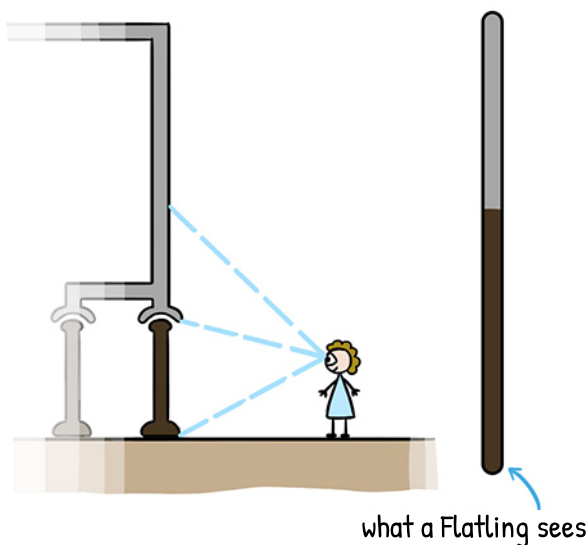
In their vision, the Flatlings, along with their buildings and natural wonders, are all reduced to lines of varying color and brightness, offering a one-dimensional outlook on a two-dimensional world.

The nature of vision means we see with one less dimension than we exist in. In our three-dimensional world, we see two-dimensional slices of our environment. Consider a box: At first glance, we see only the side facing us, not the full extent of the cube that exists. Only when we rotate the box in our hands can we grasp the entirety of its three-dimensional nature. Likewise, we can't see the entirety of a ball without rotating it and stacking that information to form a 3D mental map. It only takes a flat two-dimensional

canvas to paint a scene from our world that resembles our own vision.

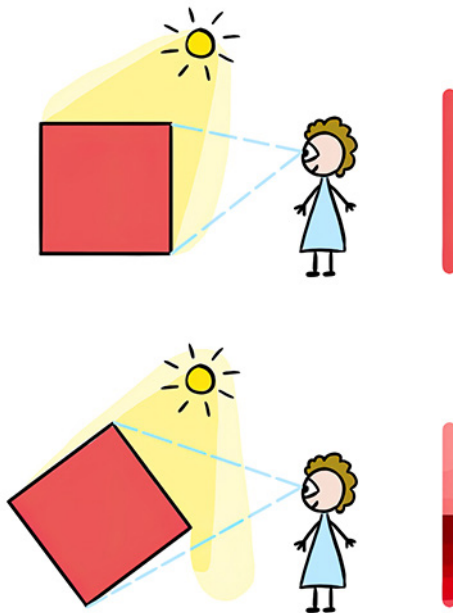
The gaze of a Flatling is similar to what the child standing at the edge of the table would see if they turned their head to the side. The facade of a grand building manifests as nothing more than a steadfast vertical gray line. A brown portion of the line represents a door, and a fleeting darkness interrupts the brown portion whenever the door is opened.

A Flatling's vision is a vertical slice so remarkably thin that it teeters on the edge of nonexistence. In the following illustrations, this perspective has been thickened a little, just to make it visible on the page!



A flat sun provides light to Twosville, which helps a lot with vision. Even with just one eye, the Flatlings are usually able to discern shadings and differences in illumination of shapes, as well as shadows on the ground that all offer clues about the orientation of objects and their distance from the viewer.

Take, for instance, a square. Suppose it is orientated such that it is flat against the ground. In that case, a Flatling will see it as a short, solid line. But, if the corner is oriented towards them, a shadow is cast on the lower side, highlighting the corner's position. Brightness changes are indicators of an object's orientation in space, clues to the layout of the 2D landscape.



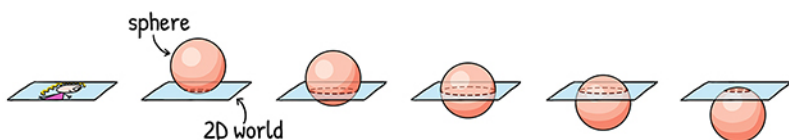
The Flatlings are aware that one of their bodily shortcomings is the lack of multiple eyes. If they had a second eye, they would get the benefit of some depth perception. This would work much like how our own two eyes give us the ability to perceive depth. The slight differences in the images from each eye give our brains the ability to judge how far away something is. Because of this, bionic eyes are something the Flatlings are researching with much passion.

Perhaps on a future visit we will one day see Flatlings carrying extra eyes to help them navigate their world.

THE GIRL IN THE PINK DRESS

As we get acquainted with our new perspective, our first companion appears. Our viewing window is filled with the vast expanse of pink from the front of her dress. Moving up and down, she appears to inspect our ship, trying to make sense of its shape. From her perspective, our vessel initially resembles a tall line, but as she looks at it from various angles, she gets a sense of its curvature and can deduce it to be a circle.

3D object passing through 2D world

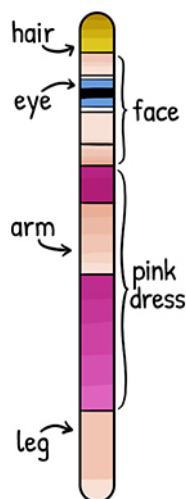


What a Flatling sees



Vessels from the third dimension do occasionally pass through Twosville. Ours would appear as a small circle, grow to maximum diameter then shrink again before disappearing completely once we were on the other side. Any Flatling that witnessed it would merely remark how strange it was that a series of many differently sized shapes appeared and vanished. The concept that an infinite number of circles could stack together in a third dimension to form a **sphere** is totally alien to them.

A closer look at our newfound companion reveals the intricate geometry of her form. A bright segment hints at the corner of a triangular dress, while a dash of purple indicates the presence of a cap—a semicircle resting gently on her head. I realize I have met this girl before, which explains why she approaches us with curiosity rather than fear. Her name is Tudy, and her presence will be an excellent way for me to demonstrate the art of making friends in Twosville.



Making friends here requires a certain kind of charm. The usual strategies of a warm smile or a firm handshake simply do not translate. Yet, there's something universal about kindness and curiosity that seems to cut across dimensions.

Keep in mind that the communication we take for granted back home is on precarious ground in this 2D world. You are no longer three-dimensional and complex, nor will you be perceived as such. To Tudy, you exist merely as an assortment of lines—a one-dimensional sight in her two-dimensional plane. And so, we face an unusual quandary: How do we foster friendship when we're perceived as no more than an ambiguous squiggle on the canvas of existence?

Your approach must be delicate, as our usual 3D movements translate into dramatic 2D transformations Tudy might perceive as alarming. So, when you are ready, make your presence known with patience and respect: Slowly open the window and gently wave your hand.

Nicely done! I see Tudy's response in the subtle shift of her form. A short line segment growing in prominence is her

finger extending towards us—a simple yet profound gesture in this dimension. Extend yours as well to meet hers halfway. And there you have it: Making friends in Twosville is not so hard, after all.

Tudy has brought along an object, possibly a token of her hospitality. Watch her closely. She's lifting something—a peculiar artifact with solid boundaries. As she manipulates it, the object's defining features come into view: four corners, tightly sealed, completed with a miniature golden lock and a hinge. It's a box.

Tudy speaks, but alas, we cannot understand her. I'm still mastering the Flatling language, and these particular sounds elude my understanding. Perhaps it's an attempt to enlighten us about the box's contents. I'm intrigued to know what's inside.

Now, I have a strategy to find out. Let's go back up to the rooftop of our vessel. There we regain our bird's-eye perspective. We can see our new friend from above. She sees our faces vanish from the window, but with no concept of upwards, she could never know where we have gone.

Suddenly, a glimmer catches my eye. Concealed within the box, now openly visible from our perspective, lies an exquisite, multifaceted gem reflecting an array of brilliant colors. It's an honor to witness such a prized possession that would otherwise have been impossible to perceive within the plane of Twosville without a key.



Before we leave, it would be nice to give Tudy a keepsake of our memorable encounter across dimensions. From here, we could easily place an extra item into her safe for her

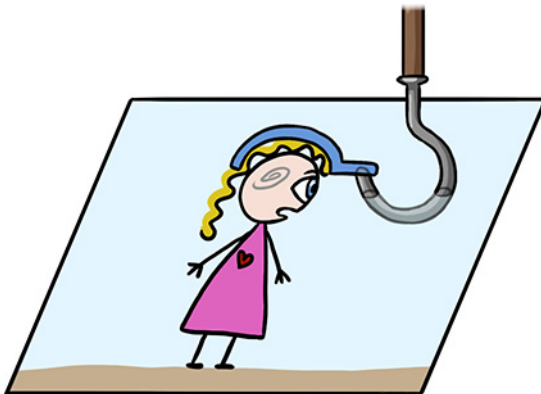
to find and be delighted by later. Do you have anything to spare? Unfortunately, the only things I have in abundance on this ship are socks.

Oh, you're fresh out of gifts? That's okay. And you're right, it might be a little freaky to find something you don't own appear inside of your locked safe. No worries, I have a different kind of keepsake in mind. One that we can make for Tudy in no time.

THE BACKWARDS CAP

Floating above the flat world, we look down to see how fashionable our new friend is. Tudy's hat is as flat as the rest of her world, a mere line curving over her circular head. But not for long, dear friend.

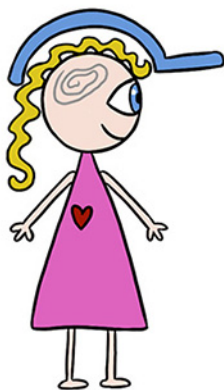
Conveniently, I keep a tool aboard the ship, primarily intended for the rare occurrence of overboard rescues. It's a simple wooden pole with a hook at its end. As we tower above the 2D plane, I can extend this pole and hook it onto the front brim of her cap. Since the pole enters the 2D world from above, she doesn't see it coming. When the hook is in her world, Tudy will see it as a round spot, a small **ellipse**, or maybe even two spots, depending on how deep the hook is into her world.



Got it! With a careful upward motion, the hook flips the brim around so that it faces backward on Tudy's head. And with this simple maneuver, an ordinary 2D hat has been transformed into a three-dimensional keepsake of a visit from a dimension beyond.

That might not seem so impressive until you truly realize what I've done. From above, we now see that the hat is backward on Tudy's head. This could not have been achieved without us. No matter how many times a Flatling tries, they cannot turn a hat backward. If they attempted, it would merely rotate until it was sitting upside down on the top of their head like a fruit bowl, with the brim sticking out the back.

original



2D attempt



our gift



Mirror images can't be achieved through rotation. A left-sider could never become a right-sider unless flipped through a higher dimension. Our flip took the cap out of Tudy's 2D world momentarily, reducing it to a vanishing line for a brief moment. Then it reappeared, stylishly backward.

Awestruck, Tudy reaches up behind her head and taps her hat's brim. She takes a few steps backward toward someone calling her from behind. Something like a laugh is heard as the Flatling behind realizes that Tudy's eye is on the other side of her head.

As our feline companion, Lemon, settles back into her perch, it signals our departure. Our farewell wave to Tudy is also goodbye to this 2D world. But we have left our friend with a precious gift. Tudy might be a little confused at first, but with time, she will realize just how precious it truly is: a playful relic of a higher dimension.

THE STOLEN GEM

As we begin to leave, our vessel casts a shadow on Twosville that is as perplexing to our 2D friends as their flat world was to us. But with all shadows come darkness, and beneath us, I hear joy turn to horror. A laugh becomes a shriek.

Look back with me now: What can you see? There is Tudy. She is looking inside her box. From up here, we can see that the box is empty. She looks at a solid brown line without the dash of color she expected, and we see the four sides of a hollow square. **Hyperspace** help her!

A thief from our own third dimension must be meddling in the affairs of the flat world. I trust you all, so I will not accuse anyone on this ship of such a crime. Perhaps a boat full of pirates is out there sailing our seas. We should keep an eye out. To Tudy, this is an event of cosmic proportions, a disturbance from a plane of existence beyond her understanding.

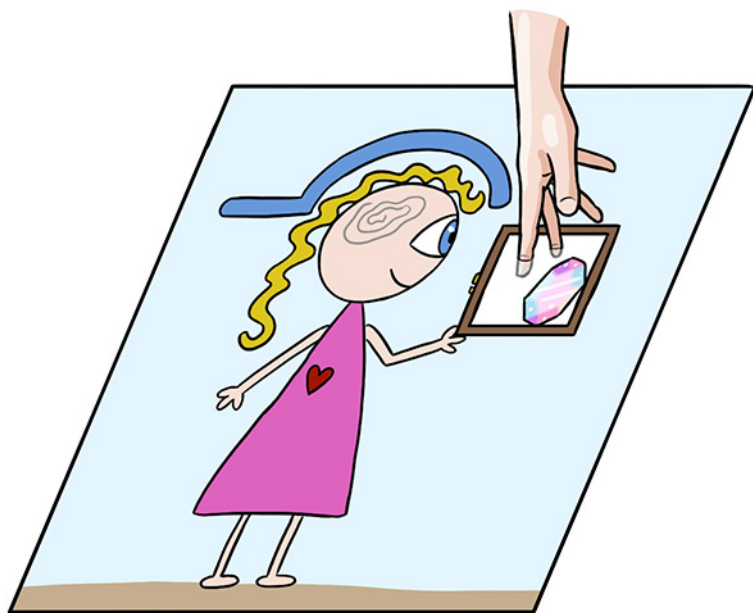
before



after



As 3D beings, we can interact with the 2D plane in ways that are incomprehensible to its inhabitants. Consider the gem, safely enclosed within its 2D safe. To us, it's an open book, visible and accessible. A 3D thief wouldn't have to "break" into the safe, nor even "intersect" with it. They would simply pick up the gem, lifting it in a direction that doesn't exist for Tudy—a concept as hard for her as attempting to imagine a new color is for us.



For her, the gem would simply disappear, as if by magic, gone without a trace. There would be no sign of intrusion, no evidence of a crime, nothing but an empty safe and a world of bewilderment. It is a peculiar kind of violation that leaves no mark and no broken boundaries—just an inexplicable absence. In our usual 3D world, it would be as if someone appeared in a closed room or took the yolk from an egg without breaking the shell.

It is so unfortunate for Tudy, but, in a way, lucky for us to witness. It demonstrates starkly how the rules of reality change as we traverse dimensions. As we observe Tudy grapple with her loss, we see a reflection of our own limitations. For who knows, perhaps there are higher dimensions beyond our comprehension, where beings could interact with our world in ways as baffling to us as this event is to Tudy.

We have no walls up against the **fourth dimension**, but as soon as we figure out how to build them, that's the first thing we should do. I hope this theft hasn't disturbed some interdimensional truce. The fourth dimension is as vulnerable to the fifth as we are to the fourth.

MOBY MENDY

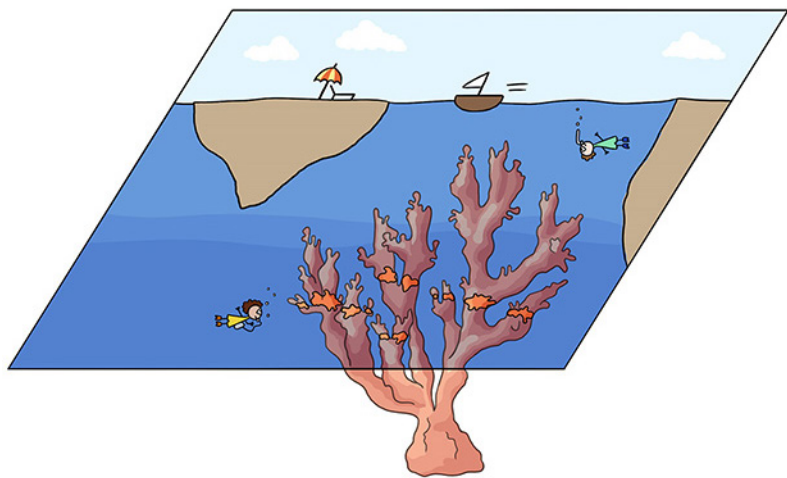
I feel like we owe it to Tudy to try and help her find her gem. We were some of the last people to see it, after all. I want you to keep a keen eye fixed on the city below while I steer the vessel around Twosville. Look inside every house and every closed box. This will be much easier for us than for any investigators on the ground. They would have to knock on each door, whereas we merely float by, seeing everything.

Apologies to any Flatlings currently bathing or otherwise seeking privacy. Unfortunately, there is no privacy from the third dimension. Let's not think of ourselves as any better off, though; we, likewise, have no privacy from the **fourth dimension**. And it's not just that a four-dimensional creature might be able to see inside our bathrooms. The privacy invasion is much more disturbing than that. The skin of our bodies, like the walls of our houses, would offer no protection. A four-dimensional creature could see inside our bodies, watch our hearts beating, and see every neuron in our brains.

Look closely as we pass over Twosville. You can actually see Tudy's heart beating quickly as she starts door-knocking for help. With a keen eye, you will notice that her pink dress is not a dress at all by our standards. Instead, the shape of her body itself is contorted into a triangle and dyed pink. I hope she is not too uncomfortable in the name of fashion.

I've seen lots of interesting stuff tucked away in homes, but no gems that look like the one Tudy had. Any luck on your end? We are approaching a beach where the ground turns to water. There are some free-floating islands suspended in the water and boats sailing between them. The boats are designed to be symmetrical so that they can get both to and from the island since they won't be able to turn around.

I see a Flatling scuba diving below an island with a tank of air. They are looking at the squiggles of two-dimensional coral. From where we are we can see something interesting: The coral that the diver is examining is actually the protrusions of a larger lump of three-dimensional coral stuck in the 2D plane. The disconnected pieces of coral in the plane are joined like the fingers on a higher-dimensional hand.



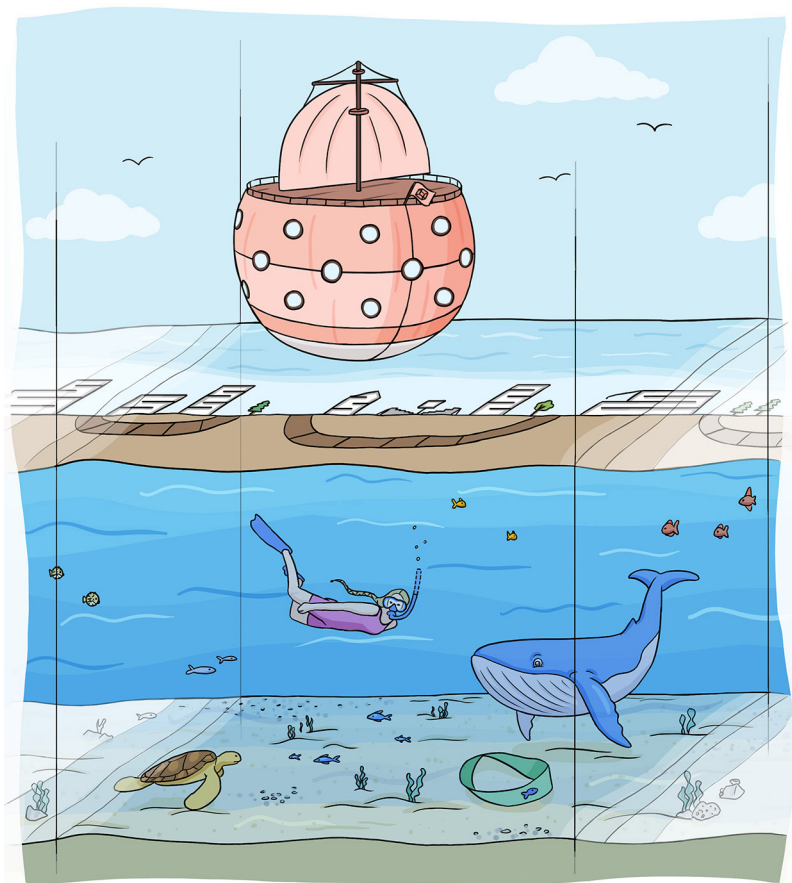
The diver can carry a roughly 30-minute supply of air. When that gets close to empty, they need to return to the surface. Little do they know there is another source of fresh air very close to them at all times down there. If only they had a snorkel that could extend out to the third dimension, they would have a constant supply. We'd have to hope that the straw could also convert our three-dimensional air into a form compatible with their two-dimensional lungs.

That gives me an idea: Maybe we should go scuba diving! I could fasten a straw to extend out into the fourth dimension, and we would be able to breathe underwater for as long as we wanted. We are currently sailing on top of a vast 3D ocean, and the 2D world is just a thin membrane on the surface between air and water. If we dive beneath the 2D world, beneath the surface, we might be able to find the gem, especially if it was stolen from below rather than from above.

Get your swimmers on! If the gem thief is down there in the depths of the ocean, we want to catch them before they get too far away.

Here's your breathing straw. When it's properly fitted, you can't see most of it, only the part that starts in the third dimension and connects to your mouth. The stalk sticks out into the fourth dimension, in a direction invisible and unimaginable to us. Be careful down there, and for goodness' sake, please don't suck up any four-dimensional bugs into your lungs, although I'm sure they'd have no problem finding their own way home.

As you get used to breathing four-dimensional air, let's say hello to the barnacles on the underside of our ship. Each barnacle lives in its own little zero-dimensional world. They have far less freedom than even the 2D Flatlings and yet still seem to make the most of existence.



We've been met with silence. I don't take offense to this, it's simply the 0D way of things. They are not aware of any dimensions at all. I do try to stop by and explain the way of things to them, however, with the hope that they might be able to at least hear and comprehend. But this is unlikely: To a creature of zero dimensions, the only thing that exists in the universe is themselves.

Follow me as we descend into the water. If you turn and look back up, you will see the 2D world again, but this time we

are viewing it from the other side. We are seeing the mirror image of the world compared with how we viewed it from above. Swimming around us are fish, turtles, and floating chunks of seaweed. With the water making us feel much lighter than on land, we can move with ease in every direction, even up and down. This is a taste of 3D living at its finest.

You might have guessed from the speed at which I fastened your breathing straw that this isn't my first rodeo. And by that I mean it isn't my first time scuba diving down here below Twosville. I do sometimes bring special passengers down here, especially if it's their first time on an interdimensional cruise.

I do have a friend down here I'd like you to meet. I hope she's around, because she's also the best person to ask if any gems have been spotted. She has a very keen eye for small objects. Her name is Moby Mendy, and she's a humpback whale.

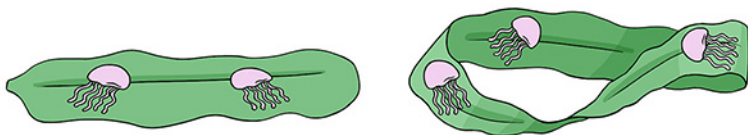
Moby is a regular three-dimensional whale. The way she spends her days, however, is far from ordinary. She is very charitable, always helping others—a true gem of the sea.

Fishing boats, stray anchors, and lost tourists all pose a danger to Twosville's sea life. Two-dimensional whales, fish, octopi, and jellyfish have all been known to suddenly disappear from the Twosville plane of existence, displaced by higher-dimensional objects. When fishing boats attempt to hook fish from Twosville, they quickly realize that 2D fish provide no sustenance in 3D, and throw them back into the ocean.

One day, Moby came across such a dislodged character by chance and had a great idea. She placed the tiny, flat sea creature upon the surface of a piece of kelp that was nearby. This allowed the 2D creature to live on a new, tiny plane of existence while embedded in the 3D waters. Moby felt sorry

for it, though. The creature was restricted to living on just one side of a piece of kelp, and she watched it moving back and forth along this finite surface all day.

Moby's most genius idea was to twist the strip of kelp and fuse the ends together, creating a loop known as a **Möbius Strip**. The single twist in the kelp allows the 2D sea creature to move forward continuously, using the kelp as an infinite treadmill. It's a major upgrade from swimming back and forth.



These days Moby keeps an eye out for any dislodged 2D creatures and provides them with a new home on a strip. She is also very handy and can mend any torn kelp or wound, which is why we call her "Moby Mendy." Speak of the devil, there she is now.

"Hello Moby! Have you seen any gems fall this way? I believe there is a pink gem missing on the surface."

Moby shakes her head and, alas, it appears that the gem hasn't fallen anywhere down here. I'm sure Moby would have seen it if it had.

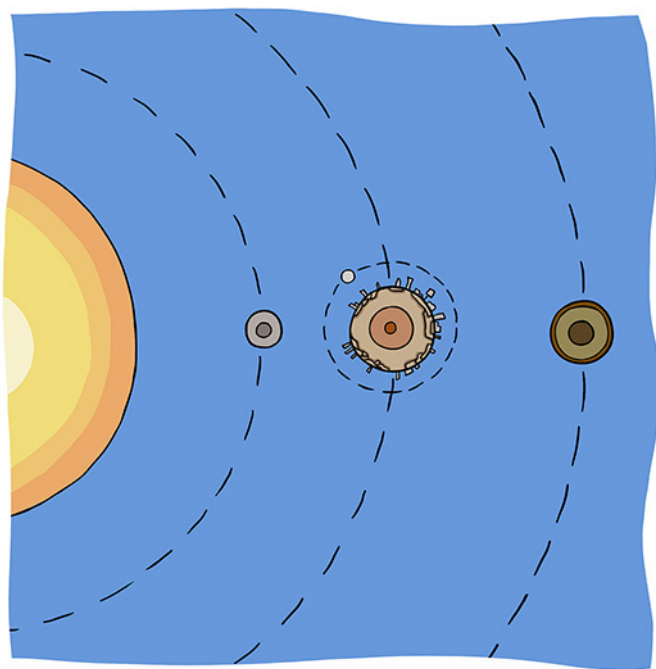
There are plenty of other interesting sights to see down here. Before we go back up, let's look at the 3D coral reef. The corals grow in all kinds of peculiar ways. Some have shapes like flowers; others zigzag or spiral around. Some corals have long tentacle-like structures, and others are round blobs, but they all seem to come together perfectly to build up an expansive network of reef.

The coral reef is a lifeform of its own, interconnected in ways that we don't fully understand. I sometimes wonder

if the seemingly individual growths are part of something larger in the fourth dimension.

Back on the surface and toweling off onboard our vessel, we seem to have exhausted all our options. We searched the depths of the oceans and inside the buildings of Twosville. We failed to find the missing gem. We also failed to find the edge of the 2D world. We traveled along to the beach, past the islands, and then found ourselves back where we started.

ZOOMING OUT

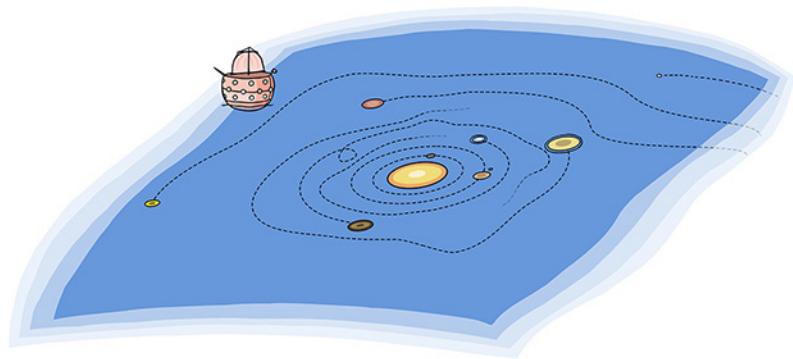


Zooming out from Twosville as we sail away, we see a bit more of what that world contains. Close up, it seemed like the land of Twosville extended endlessly to the left and right, but now we see that it is actually a large circle, forming the surface of the Flatlings' **disk**-shaped planet. This is similar to

our own Earth, which feels flat at a local level but reveals its spherical nature when viewed from afar.

The Flatlings' solar system, too, is realized in two dimensions. Their Earth, a grand disk, is orbited by a small, dot-like moon. Other planets, each in near-circular orbits, revolve around the largest disk of all: the flat sun.

Unbeknownst to our Flatlings, their entire universe, with all its stars and celestial bodies, exists on the surface of a vast ocean in a third dimension. This ocean curves around a 3D Earth in ways they could scarcely imagine. Their world, seemingly flat and endless, is but a tiny part of a much grander, multi-dimensional universe.



Here we get a chance to ponder how the laws of physics are different in an imaginary world of only two dimensions. Gravity, for one, would behave quite differently. It wouldn't follow the familiar inverse square law, where its strength fades quickly with increasing distance from its source. Instead, gravity would follow an inverse linear law, weakening at a slower rate as you moved farther away.

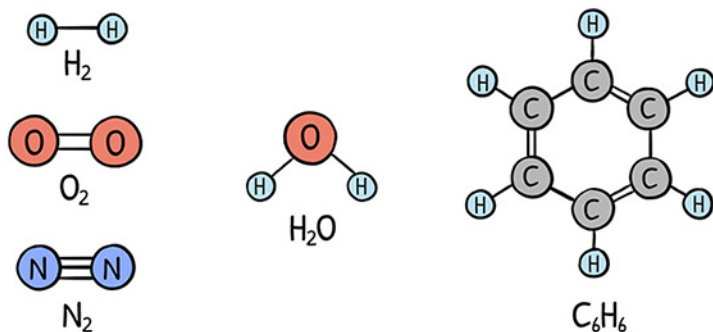
This change may make elliptical orbits, like the paths planets take in our 3D world, as unstable as a stack of dishes ten feet tall. In 2D, celestial objects might trace unpredictable swirling paths rather than stable orbits.

The state of Twosville, as I have described it, is only one of the endless imaginable possibilities. Your own mind may imagine a different version of 2D life. It all depends on which laws of physics you politely ask to stay at the party when pondering the feasibility of life in such a world.

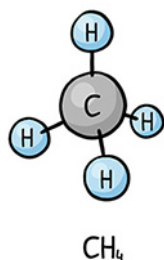
It's not just the laws of physics that need reimagining, but also the laws of chemistry. Nothing in our world exists without atoms, each one a 3D cloud of electrons around a nucleus of protons and neutrons. Could atoms exist in 2D?

If so, then the atoms that formed would be simpler than in 3D, with electron shells filling up faster than guests at a free buffet, but they would not completely lack structure. Atoms might still come together to form molecules, but only ones that can remain flat.

Simple molecules made up of just two atoms, like hydrogen (H_2), oxygen (O_2), and nitrogen (N_2), could be possible, as their bonding can exist in 2D. Water (H_2O) has a simple bend and would also be possible. So would something like benzene (C_6H_6), with its hexagonal ring structure.

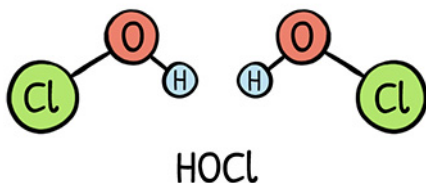


However, molecules requiring 3D configurations, such as methane (CH_4), would not be stable in 2D. Methane's tetrahedral geometry cannot be accommodated in a flat 2D space. Diamonds could not exist, as that would require



the arrangement of carbon atoms in a 3D lattice, but graphene could exist, which arranges carbon in thin sheets. Large biomolecules like proteins, which fold into intricate 3D shapes, would be severely limited. That's one more thing to keep in mind when considering the biology of Flatlings.

A Flatling chemist would have to pay more attention to the arrangement of atoms in a simple molecule than a 3D chemist. That's because, without the ability to rotate through the third dimension, some symmetrical pairs of molecules would actually behave differently from each other. Take, for example, three different atoms joined to make a simple bent molecule. An example of this is hypochlorous acid, with one hydrogen (H) atom, one oxygen (O) atom, and one chlorine (Cl) atom. This acid is often used as a disinfectant.



In our world, it doesn't matter if the hydrogen atom binds to the right or left of the central oxygen atom, because it can always be rotated to be equivalent. However, with one less plane of rotation available in 2D, the right and left arrangements are different and will have different chemical properties. Each would have a different set of other molecules that it could possibly interact with. We will meet this concept, called **chirality**, again in Part 3.

LAWN BOWLS

Now that we've left Twosville, I wish to pose a question: Can you think of any experiment that a 2D Flatling could perform to find out if there is a third dimension out there?

I met one of my best friends through our shared love of experiments. Her name is Susan, and she is a lawn bowler.

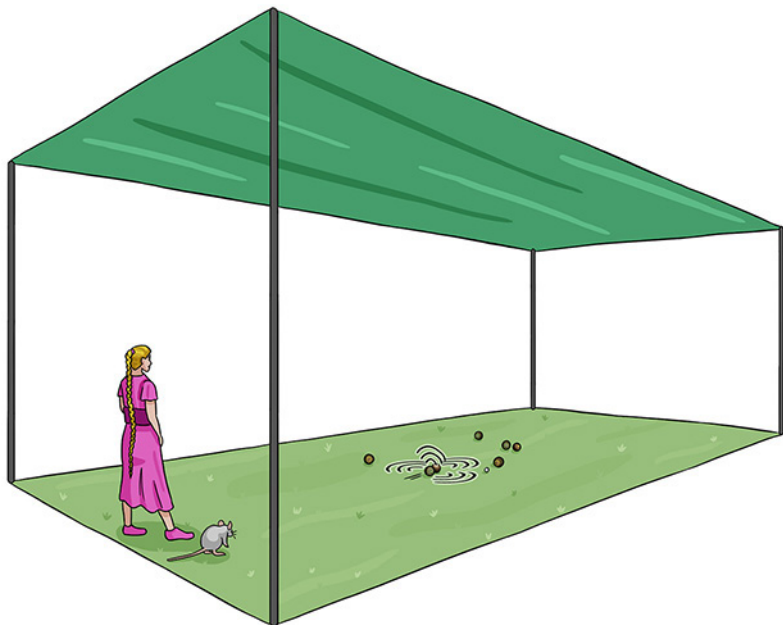
Let me tell you the story of how we met. It was a sunny afternoon in the third dimension, and I was out playing lawn bowls. It's a game that involves rolling heavy balls along a flat grass rink toward the kitty, which is a small white ball. Bowls that come to rest closest to the kitty score points.

Often, the bowls collide with each other, and when they do, there's a distinct clank. The sound comes from the energy of motion being converted into sound and heat. When any two objects, including particles, collide, energy will be conserved. Energy can take many forms, including heat, sound, light, and even matter itself. We expect the amount of energy before the collision to be equal to the energy after, though it might change forms. It's one of the unyielding laws of physics.

This day I was paired to play against Susan, and one thing to know about Susan is that she is a small mouse. This gave her the advantage of being able to quickly scurry around the bowling green to check and measure the distance to the kitty before playing her bowl. She is a formidable opponent.

During a five-minute tea break, I was talking to her about the laws of physics. We agreed that the bowling green was a great place to set up an experiment to test the conservation of energy. We just needed to add up the energy of the system before and after a collision and check that it was the same.

Susan and I each set up our own detectors to measure the energy given off by the collision and planned to then



compare our two independent results as a way to gain more confidence in the answer. I set up sophisticated sound, heat, and light detectors all around the rink, as well as on the overhead shade cloth and suspended in the corners of the rink.

Susan, however, due to being a small mouse, scurried around on the grass, only placing detectors on the ground, on the plane level with the bowls. She figured that's where all the action happens, and so that's where all the energy must be contained. From her perspective, the world is smaller and flatter than from my perspective. Compared to us, Susan lives in an approximately 2D world.

Susan scurried around with a notepad, jotting down the energy levels before and after a collision, and I did the same with my detectors. Despite our best efforts, the results did not add up when we compared numbers. Susan found that

energy wasn't conserved. Her number for the energy of the system was higher before the collision than afterward. I, however, found that they were almost equal.



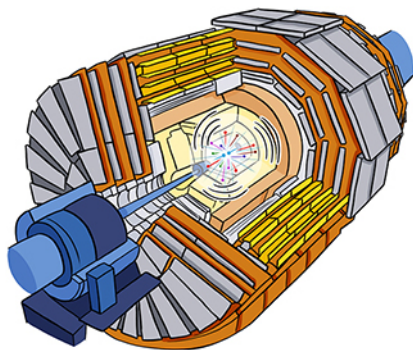
By lifting up Susan with my hand and raising her to the shade cloth to show her my reading up there, I was able to show her where she had gone wrong. Susan had only measured the energy that stayed on the flat, pseudo-two-dimensional surface of the bowling green.

Some of the energy from the collision also spreads upward, into the third dimension, where my shade cloth detector measured it. Despite her best efforts, Susan's numbers couldn't possibly add up because she was unaware of the energy escaping upward.

If some forms of energy are able to travel between dimensions, then the 2D Flatlings could design a similar experiment. Missing energy might clue them in to the existence of a third dimension. Yet, there's a lurking question: What if, as with Susan's results, there's energy escaping our detection, slipping into a dimension beyond our instruments' reach?

Scientists in our world are trying to perform experiments similar to the bowling-green test. Using particle accelerators, they keep track of the energy before and after a collision of particles and make sure it adds up. Sometimes particles break apart and change into hundreds of new particles in the

collision, but so far, experiments haven't shown any missing energy. Scientists keep a close eye on these collisions, though, especially any concerning **gravitons** (the particles thought to carry gravity), in case they might be escaping to a fourth dimension and taking some energy away with them.



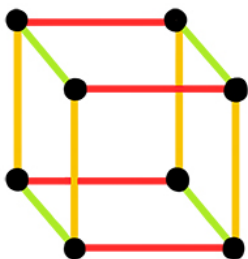
FLIP

Sometimes we come face to face,
With our mirror selves in a mirror space.
They look like us but out of place,
These doppelgangers from hyperspace.

Resembling an evil twin,
Their uncanny, familiar grin,
Makes our heads spin;
They act like they've been flipped from within.

With beliefs that we find treacherous,
Their very existence is sus,
As if the universe is ambidextrous
and made opposites just to mimic us.

Sometimes it takes moving in the direction of
our inside,
And seeing the heart of the divide,
To understand the other side,
And let them be our mirror guide.

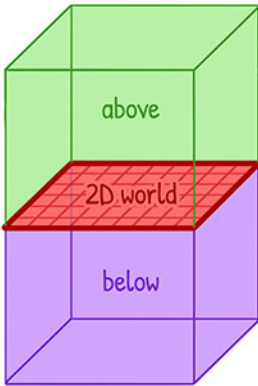


PART 3:

THREESVILLE

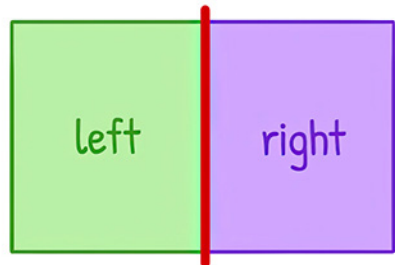
Instead of staring down at the sea, rub your eyes and take a look around you. See the seagulls flying overhead and a dolphin's tail piercing the water's surface. They are embracing depth, reminding us we live in a world of three **dimensions**. Welcome back to **Threesville**.

See that rope you are holding? Well, look beneath your grip and you will find a knot. There are knots all along this railing rope as a way to remind us where we are. There is no way to knot a rope in **Twosville**, since the rope needs to pass in front of itself and needs the third dimension to do so.



Dipping our metaphorical toe into Twosville revealed an interesting phenomenon: the dividing of our world into two distinct parts. The sky above the surface and the ocean depths below exist as separate entities, split by the two-dimensional plane we're sailing on. As our ship glides through the tranquil waters, we can appreciate both of these worlds. It's not just from above that a **3D** being can observe the flat **2D** world, since we also saw it from below. Both directions are equally inconceivable to a **Flatling**.

This principle extends even further. If a plane splits a 3D world in half, then a line splits a 2D world in half. If a straight line were laid down near Twosville's town



square, then left-siders and right-siders would be separated by a wall, destined to live in entirely separate realities, unable to ever cross over to the other side.

Here, I'll pose an intriguing question: Might our 3D world be doing the same to some higher, **fourth dimension**, dividing it into two distinct regions? A tempting analogy might be the religious notions of Heaven and Hell.

Our main focus remains grounded in playing with mathematics. So, let's assign terms to the hypothetical regions that Threesville could be cleaving. We'll call the **4D** direction "above" us **ana**, and the 4D "below" us **kata**. This way, we can conceptualize Threesville as a 3D space that separates 4D space into ana and kata. (These terms aren't of my invention; they were first coined by Charles Hinton, an early pioneer of higher-dimensional studies. They come from the Greek words meaning "up toward" and "down from," respectively.)

Our exploration so far serves merely as a stepping stone, a primer for our main voyage: a tour of the fourth dimension. There, we'll encounter new friends living in ana and kata. To get there, we can't just sail up, down, left, right, forward, or backward. Instead, we are going to travel ana. This directional move, quite unlike anything experienced in our 3D world, is so unique that it sets us apart from all other oceanic cruises.

As we navigate the passage, I will share with you three imaginative stories from Threesville, tales that I was fortunate to learn directly from a handful of intriguing characters. Each of these stories, while rooted in the familiar 3D world, offer hints of, and insights into, the mysteries we may encounter as we travel *forth*.

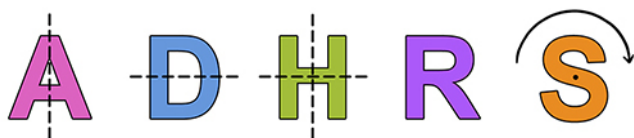
TWO LEFT FEET

In a charming corner of the world lives a young girl named Ada. The townsfolk often tease her, saying she dances with “two left feet,” as she is somewhat uncoordinated and prone to stumbling over her own shoes during her dance practice. But even with her clumsy moves, Ada loves to dance more than anything else and dreams of gliding like a professional ballerina.

I was lucky enough to visit Ada in her hometown. I was captaining this ship on a trade voyage, back when I was part of the interdimensional fruit trade. Word of Ada’s struggle made its way to me, and I became interested in helping.

At her home, I took time to examine Ada’s pair of bejeweled ballet shoes. I asked Ada if I could borrow one shoe—the right one—promising to return it shortly. I could see how worn out it was—a sign of Ada’s persistent dedication to dance. Back on my ship, I set sail on a voyage to **Foursville**, accompanied by Ada’s ballet shoe.

An object is said to have **mirror symmetry** when it can be reflected and still look exactly the same. The letters A, M, U, T, V, W, and Y all have mirror symmetry. When you draw a line down their middle, you’ll notice the left and right sides mirror each other. Letters like B, C, D, E, and K have a different kind of mirror symmetry: If you draw a horizontal line across them, the top and bottom halves can fold directly onto each other. H, I, O, and X have multiple lines of symmetry.

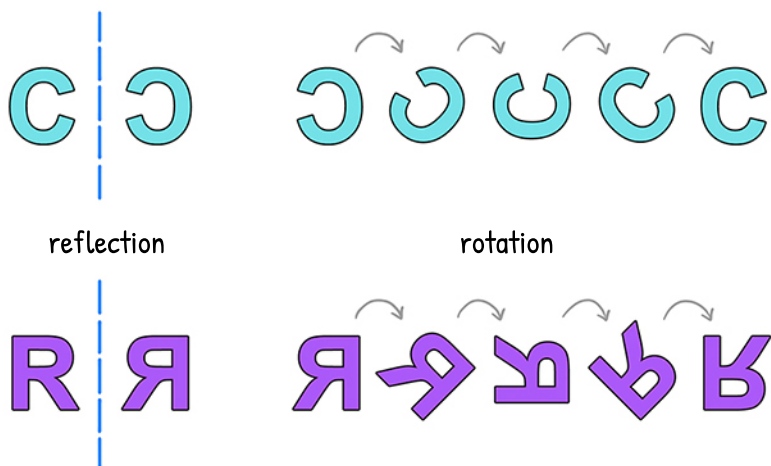


Other letters, such as L and R, don’t have mirror symmetry around any axis. Still others, like S and Z, have a different

kind of symmetry entirely: one that is rotated around a point instead of reflected around a line.

When an object with mirror symmetry is reflected in a mirror, there will be some way for the reflection in the mirror to rotate to become superimposable on the original object. This means that if the mirror image fell out of the mirror and into the real world, it could rotate to be the same as the original.

Thinking about examples of text on a page, if the letter C was reflected in a Twosville mirror (a line), a Flatling could imagine reaching a hand into the mirror, grabbing the reflected C and rotating it so it again reads C. But if an unruly R was reflected in a mirror, even if the Flatling were to grab the Я by the scruff, they wouldn't be able to rotate it to get any better than an upside-down K. Not without employing a third dimension, at least.



Humans are tantalizingly close to having a mirror symmetry across a line down our bodies, but the left and right sides of no one's face will match identically. Our internal organs are also arranged asymmetrically, with the liver located on the right side and the heart skewing slightly

to the left (unless someone has situs inversus, a rare genetic condition in which the organs in the chest and abdomen are reversed from their normal positions.) People with scoliosis have an asymmetrical curve in the spine.

Other notable asymmetric 3D objects are shoes and gloves. These objects have a quality called **handedness**. In three dimensions, no amount of rotating can make a right shoe fit a left foot. If a right shoe looked at itself in the mirror, it would have to accept that it could never become its own reflection. A glove made for the left hand cannot be worn correctly on the right hand. It's as if there's an invisible wall preventing this change, but that wall disappears if we use the fourth dimension. In four dimensions, any 3D object can be flipped in a way that creates its mirror image.



Upon reaching Foursville, I carefully flipped Ada's right shoe in such a way that it became a mirror image of itself. When the shoe was returned to Threesville, it appeared to be a perfect replica of Ada's original left shoe.

Ada was stunned as she slipped the transformed shoe back onto her foot. It fitted as if it had always been made for her. With two left shoes, Ada's dance transformed. Her "two left feet" became a strength instead of a hindrance. Everyone in the town was amazed as they watched Ada dance, her

movements now elegant and graceful, all thanks to her unique shoes.

The way that I flipped the shoe into something that it couldn't become using only three dimensions is similar to what happened when I flipped the hat of our 2D friend Tudy. I gave her a backward hat that no amount of rotating in her dimension could have achieved. In Twosville, her hat lacked mirror symmetry, but by using the third dimension, we were able to achieve a reflection.

HYPERSPHERE SOUP

Another friend that I have been lucky to meet is a man named Roger. He is a humble scholar with a fondness for mathematics and an appetite for soup. He told me about one extraordinary encounter where his twin passions collided while he was sitting in the corner of a quiet café.

Sitting in his usual spot by the window, Roger was deeply engrossed in a book on multi-dimensional physics while savoring his favorite pumpkin soup. The chapter he was reading was about **hyperspace**. Not just the mysterious fourth dimension, but the fifth, sixth, and seventh dimensions, and even beyond. The prefix **hyper**, meaning "above," alludes to dimensions beyond our tangible experience.

Shapes that exist in these higher dimensions are referred to as hypershapes. Among them are the intriguing **hypersphere** and the mesmerizing **hypercube**. Each of these shapes has properties and behaviors that defy the logic of our three-dimensional experiences.

Roger's book explained that, just as a three-dimensional object can be used to confuse hypothetical Flatlings, a hypershape can manifest in our world in ways that our minds might struggle to comprehend. Little did he know, Roger was about to witness this very phenomenon first-hand.

As he glanced into his bowl between paragraphs, his eyes widened in astonishment. His soup, it seemed, was not behaving normally. Out of nowhere, a small **sphere** appeared, bobbing in the middle of the bowl as if a droplet of another substance had been added. Bewildered, Roger leaned closer. The sphere grew in size, expanding steadily until it was nearly as wide as the bowl itself. Roger blinked in disbelief. Then, just as steadily as it had grown, the sphere started to shrink until it once more became a mere point, before disappearing entirely.



What Roger witnessed was not a trick of the light or an oddity of the soup's ingredients. He had seen a hypersphere—a four-dimensional sphere—passing through his three-dimensional bowl.

When a four-dimensional object like a hypersphere crosses into our 3D space, we don't see it in its entirety. Instead, we witness a series of **cross-sections**, much like viewing slices of a 3D object in 2D. In this case, as the hypersphere passed through Roger's soup, he saw it as an expanding and then contracting 3D sphere.

Roger was struck with a sense of awe but also of slight annoyance. He picked up his spoon, poking at the spot where the hypersphere had appeared, and found his soup unaltered; nothing had been left behind by the hypersphere except for a few ripples. The taste was as delightful as ever, and the consistency was unchanged. Yet, he couldn't shake the feeling of discomfort.

Feeling a bit disoriented, Roger looked around the room. The other diners were oblivious to the strange phenomenon that had just occurred in his soup bowl. He waved his hand to catch the attention of a passing waiter.

“Excuse me,” Roger began, in his most polite yet bewildered tone. “You might not believe this, but a hypersphere just passed through my soup.” As Roger described what he had just witnessed, the waiter’s bemusement was clear. Nonetheless, the waiter promised to convey Roger’s unusual feedback to the chef.

This day marked the start of Roger’s fascination with the fourth dimension. This once-ordinary man had had a taste of the extraordinary and developed an insatiable appetite for the mysteries of higher-dimensional spaces.

I had the privilege of meeting Roger and hearing this story first-hand last summer when he became a passenger aboard the *Vessel of Higher Dimensions*. His eyes sparkled with anticipation and curiosity. On this ship, navigating the unfamiliar waters of the fourth dimension, Roger had the long-awaited opportunity to glimpse a hypersphere once again, this time in its native dimension. I was grateful to have brought him this moment of closure.

THE WOMAN WHO FLIPPED OUT

In tales of dimensional travels, some stories stand out more than others due to the shocking events they entail. Such is the story of Doda, a brave spirit who got more than she bargained for during her trip aboard this ship. Her encounter with the fourth dimension was as mentally tumultuous as it was physically.

As you already know, this ship sails the vast oceans of imagination. Onboard, there’s an array of apparatuses and tools built to navigate higher dimensions. One

such fictional instrument is the TesserComm—a communication tool to intercept and translate messages from the fourth dimension.

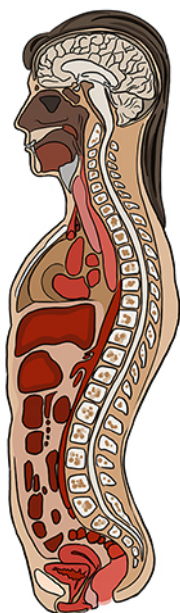
The tour started with Doda asking me many questions about the ship and our destinations in her inquisitive fashion. Some of them I couldn't answer, since I, too, am just another voyager, albeit one with more trips under my belt. Doda and her husband observed the breathtaking views together amid a sea of what-ifs.

The trip took an unforeseen twist when the TesserComm hummed to life, signaling an incoming transmission. I rushed to decode the message. It was from someone with whom I have much in common, despite living in different dimensions. It was from a Foursville tour guide who, much like me, captained a ship offering trips to observe other novel dimensions.

The 4D captain's message was a mix of caution and curiosity. They expressed interest in our human voyagers. The 4D captain proposed a mutual experiment: an exchange of experiences between our dimensions. Having had brief interactions with this captain previously, I trusted their judgment.

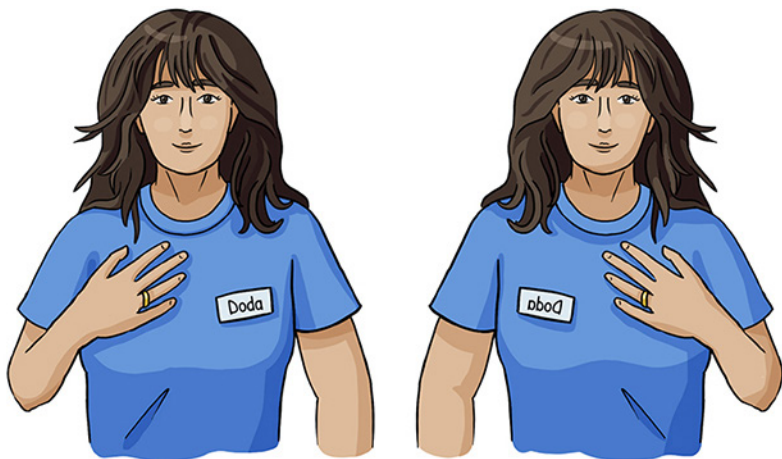
I asked for a volunteer, and Doda's hand shot up. I guided her to follow the instructions coming via the TesserComm. Doda positioned herself on the ship's deck as the other passengers and I gathered around. One passenger held up a camera to record the event. Doda saw him and called out, "Make sure you get my good side!" just before she disappeared.

With an uncanny silence that punctuated the moment, Doda was flipped through the fourth dimension. What remained in our three-dimensional world was a startling slice of her existence: a vertical cross-section of a human body.



Recall that when we flipped the hat of our 2D friend, a thin line of the hat remained in the flat plane while the rest of the hat was moving above and below that axis. In Doda's case, the rest of her body was moving in the fourth dimension around this slice of her, leaving a two-dimensional cross-section in our world. Similar to those anatomical models you might see in a museum, it was complete with bits of her skeleton, organs, and intricate networks of veins.

Most passengers were left aghast, forced to reckon with the chilling reality of higher dimensional perils. The sight of Doda, the once inquisitive explorer, reduced to a mere slice, served as a reality check. When she returned to her original form, she seemed mostly the same, except that the text on her name badge now read backward, as if being viewed in a mirror. Doda had been transformed into something that resembled pboD. Just like the hat of the 2D creature, no matter how much Doda tried to rotate the name badge, it wouldn't go back to normal.



Doda had been granted a glimpse into the vast expanse of the fourth dimension. As she'd marveled at the complex 4D landscapes, patterns, and entities, the four-dimensional natives on their own guided tour were being introduced to the fascinating concept of flipping a 3D object. We can assume that perhaps their tour guide was also helping them to imagine what it might be like for a 4D object to be flipped by a boat full of five-dimensional travelers.

But it was her husband who, acquainted with Doda's quirks and features, recognized the most unsettling changes. Her left eye, which was always a touch larger, had mysteriously migrated to her right. Her teeth, once crooked in a familiar fashion, are now curved in the opposite direction.

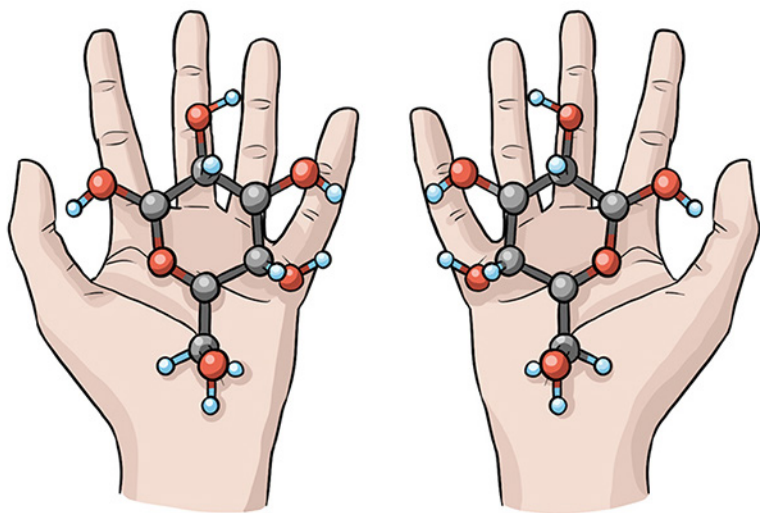
I held my tongue in that tense moment, not wanting to compound their distress, but I knew that the mirror flip had affected more than just Doda's external appearance. The layout of her organs would have been reversed, too, with her heart, usually slightly to the left, now leaning toward the right.

When mirror-flipping something, every aspect of the object would be affected. This includes the molecular and atomic structures. The implications of this are profoundly complex. Remember back to the story about Ada and her two left feet? Shoes are an example of something with "handedness."

In chemistry, molecules that have a handedness are called **chiral**. The term chirality is, in fact, derived from the Greek word for "hand." Just as left and right shoes are mirror images of each other and are non-superimposable, chiral molecules have non-superimposable mirror images. That means the mirror image of the molecule could not be rotated to be perfectly overlaid on the original.

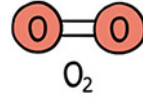
DNA, the code of life, is a chiral molecule that naturally spirals in a right-handed manner. When Doda became her mirror image, her DNA was also flipped into a left-handed form, spiraling in the opposite direction. The chirality of biological molecules, like DNA, often plays a critical role in their function. The interactions of DNA with proteins, for example, are highly specific, and a flipped strand of DNA might behave in unpredictable ways.

All of the molecules inside Doda's body were now flipped and could still work together in a relatively normal way, but an untold number of chemicals from the unflipped world outside her body would now be interacting in different, possibly harmful ways. The biological consequences for Doda would be profound. It wouldn't just be a matter of aesthetics or external changes; her entire biochemistry would be impacted.



She was still breathing okay for now. Oxygen is not a chiral molecule, so she was still able to obtain what she needed from the air. Oxygen molecules consist of two oxygen atoms

bonded in a symmetric fashion, so the mirror image is indistinguishable from the original.



However, for the new Doda, food would taste odd because the forms of some of the molecules responsible for taste were no longer compatible with her altered biology. Much more alarmingly, she might fail to gain the nutrients she needs from food. An attentive fellow passenger by the name of Andrew realized this and, in distress, pointed out that glucose is a chiral molecule and that the mirrored form of normal glucose, known as L-glucose, is not produced in nature.

Doda's flipped molecules and enzymes would need L-glucose to function, but she wouldn't be able to find it in normal food. I do not stock supplies of inverse chirality food, so I realized we were in deep trouble. Maybe if I left out some food, the 4D captain could flip that for Doda to eat?

Amid this unsettling event, there was an odd silver lining. Doda, or should I call her pboQ, who had always been bothered by the smudging from her left-handed writing, was now right-handed. For the first time on the tour, she also found that her cat allergy had disappeared. The allergens shed by my cat Lemon, which had once triggered sneezing fits and rashes in Doda, now seemed benign, unable to interact with her flipped biochemistry. Her mirrored body had granted her these unexpected yet temporary reprieves.

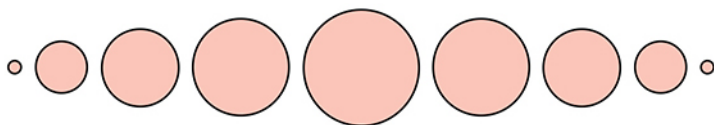
Not wanting to explain the grave nature of the predicament to Doda, I discreetly leaned back upon the TesserComm's interface and sent a mayday call to the fourth dimension, asking them to flip Doda back to normal. The 4D captain promptly agreed, and my passengers gasped once more as Doda's body became a thin slice before appearing back in its original form, a final twist in the tale of a woman who quite literally flipped out.

HOW TO DRAW A TESSERACT

Let's ponder these strange events while we continue our voyage toward Foursville. We can see many analogies between what we experienced in Twosville and what we might experience if a 4D being interacted with us.

Think about Roger and his soup. This story helps us go some way toward imagining what four-dimensional shapes will look like. Roger saw an infinitude of differently sized spheres in his soup as a four-dimensional sphere passed through his bowl. Could this help us to draw what the entire hypersphere looks like?

When our ship appeared in Twosville, our friend Tudy saw it as a collection of circles. If we told her to combine those circles she saw into a drawing of a sphere, she might erroneously draw them all lined up next to each other. She would be there all day drawing them and no closer to knowing what a sphere looks like. Similarly erroneous for us would be picturing a hypersphere by arranging a bunch of spheres of varying size in a line like a string of pearls.



Tudy needs to imagine stacking the circles in a dimension she isn't familiar with. She must stack them on top of each other from smallest to biggest and then back to smallest again. Unfortunately, she has no concept of "on top." Likewise, we would need to imagine stacking spheres in a dimension equally unfamiliar to us.

Let's try thinking it through with another shape. A one-dimensional line can be drawn by stacking zero-dimensional dots. A square can be drawn by stacking one-dimensional lines, and a cube can be made by stacking squares. Finally,

a four-dimensional hypercube, also known as a **tesseract**, can be made by stacking three-dimensional cubes.

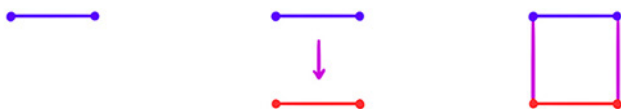
But how should we stack the cube to make a tesseract? I can't hand you a physical tesseract (for our world doesn't allow us that luxury). Still, I can certainly guide you on visualizing and drawing one. It's much like trying to paint a picture of a dream; it's ethereal and elusive, but we can still try.

Try sketching it on a sheet of paper! Give it a go now and see what you come up with. Remember that it's as if a cube has been extended in an extra direction that's perpendicular to the usual three. Trying to picture it in your head is already a challenge, and putting pen to paper might be close to impossible, but you will find it interesting to look back on your attempt. Maybe even as you think more about the tesseract, your ability to visualize it will improve. I can't guarantee ever reaching enlightenment, though.

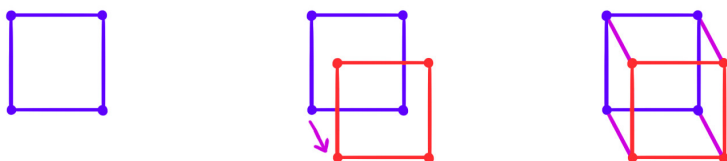
If you're feeling hyper-dimensional today, you could try also drawing a 4D hypersphere beneath your drawing of a tesseract. How could the spheres Roger saw in his soup be joined in a dimension we can't observe? Should we draw them next to each other, nestled inside each other, or some other way? Does another way to draw them even exist? If you can think of one, you might be bound for a fruitful career in mathematics.

Now that you've had your own go, let's take a look at some of the drawings that past passengers left in the guest book.

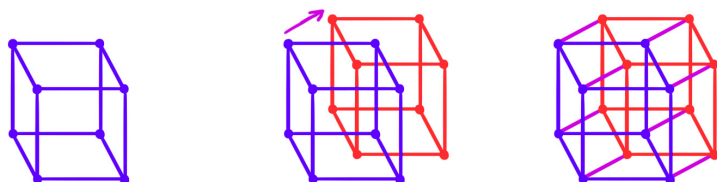
One of the most popular methods for drawing a tesseract takes inspiration from how we draw a cube. Drawing a cube on paper begins with one square.



We can then sketch a second square, slightly offset and diagonally placed in relation to the first. Connecting the corresponding corners of these squares gives us the three-dimensional representation of a cube.



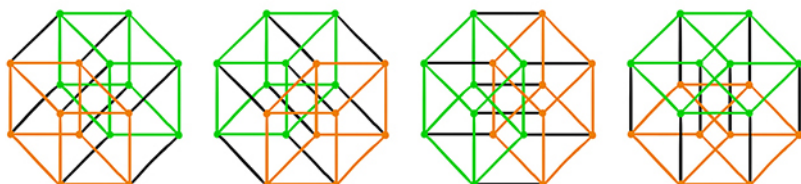
Now, to venture into the realm of the tesseract, we'll draw another cube, this time offset in a different diagonal direction.



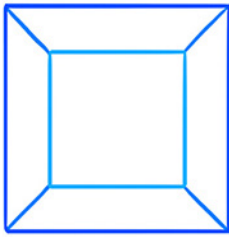
Imagining an infinite cascade of cubes between these two outermost cubes offers us one interpretation of a solid tesseract.

Studying another **wireframe** depiction below, can you discern how many cubes there are?

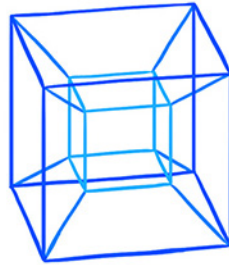
There are four pairs of cubes, each offset, making a total of eight.



Another method of drawing a tesseract comes from those who have mastered the art of perspective drawing. What if we looked directly at the front of a cube instead of at its side? If it were transparent, we'd observe a prominent square closer to us, with its rear counterpart appearing slightly smaller within it. Applying this principle with hypercubes results in a smaller cube nestled within its larger counterpart.

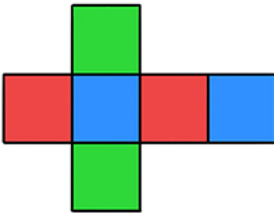


cube

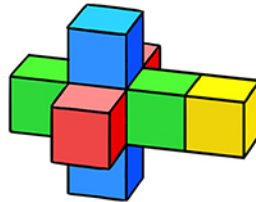


tesseract

Not all interpretations are bound to lines on paper. The more practical, hands-on explorers might draw inspiration from the common practice of crafting a cube from a two-dimensional net. One can extend the idea into the fourth dimension. Here I present to you a net that, when appropriately folded along its eight cubic faces, aspires to form a tesseract. If you're up for a delightful mental challenge, try to envisage this folding process and the resulting hypercube.



cube



tesseract

JUNGLE GYM

A memorable past passenger who thought much about visualizing the fourth dimension was Charles Hinton. He invented the name *tesseract* for the 4D cube. He took inspiration from the Greek words *téssara* ("four") and *aktís* ("ray"), describing the four edges extending from each vertex to the other vertices. In his 1888 description of a tesseract, he asked us to "suppose there is an unknown direction at right angles to all our known directions, just as a third direction would be unknown to a being confined to the surface of the table." Moving a cube in that new direction will trace out a tesseract.

Hinton spent some time as a mathematics teacher and was intrigued by how he could teach children about the fourth dimension. He himself had spent a lot of time trying to visualize four-dimensional shapes, with limited success, but he thought children might have an advantage in this pursuit due to their growing minds. These budding brains might be able to take on new ideas more easily, similar to how children generally find it easier than adults to learn new languages.

Hinton thought that adults might have a hard time grasping the fourth dimension because they don't even have an intuitive grasp of the third dimension. We spend most of our time moving around on mostly flat ground. Even when we climb over an obstacle, it is nothing compared to the true three-dimensional experience that a bird or a fish may have. To a bird, the entire sky is at their disposal.

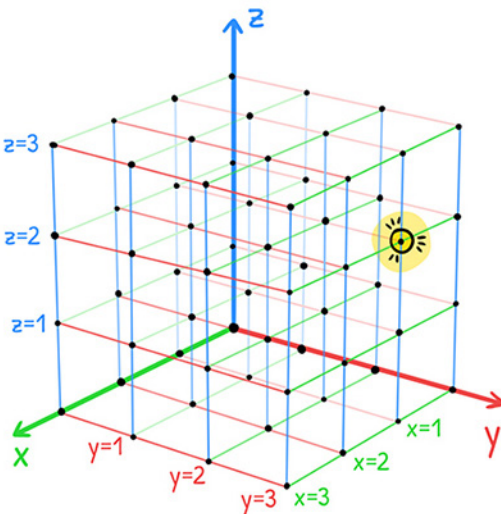
We would have a hard time adjusting to the ability to move freely in three dimensions, so it's no wonder that we struggle to make the additional jump to imagining four dimensions. Hinton wanted to expose children to true three-dimensional movement, so he built bamboo climbing structures named with the three axes: X, Y, and Z.

Hinton's son Sebastian was inspired by these structures and went on to invent the jungle gym, the lattice-like tower seen in playgrounds all over the world. In his patent, he wrote, "I have designed a climbing apparatus, so proportioned and constructed that it provides a kind of forest top through which a troop of children may play in a manner somewhat similar to that of a troop of monkeys through the treetops in a jungle."

Now here's your chance to win a prize. We will play some games on the jungle gym that we have here on board to help you build up your intuitive grasp of three dimensions. I want three of you to each take a corner at the base of the structure. I'll call out the name of a point on the structure, and the first one to get there wins a prize.



Ready? X1, Y3, Z2, go!



Dearest passenger, you will need to go faster than that next time. Don't forget you can move through the diagonals.

Let's try another, from where you are: get to X0, Y0, Z5!

Wow, look at you up there on the top corner of the frame.

It's a great point to get a view of the water, don't you agree?

Hinton's structures might not have given children the ability to think in four dimensions, but they sure were fun.

PUPPET SHOW

Remember the shadow-puppet show that we witnessed during our visit to Twosville? It was fun to watch the little Flatlings' awe at moving lines cast by flat shapes. Well, I've got a little surprise for you onboard this ship.

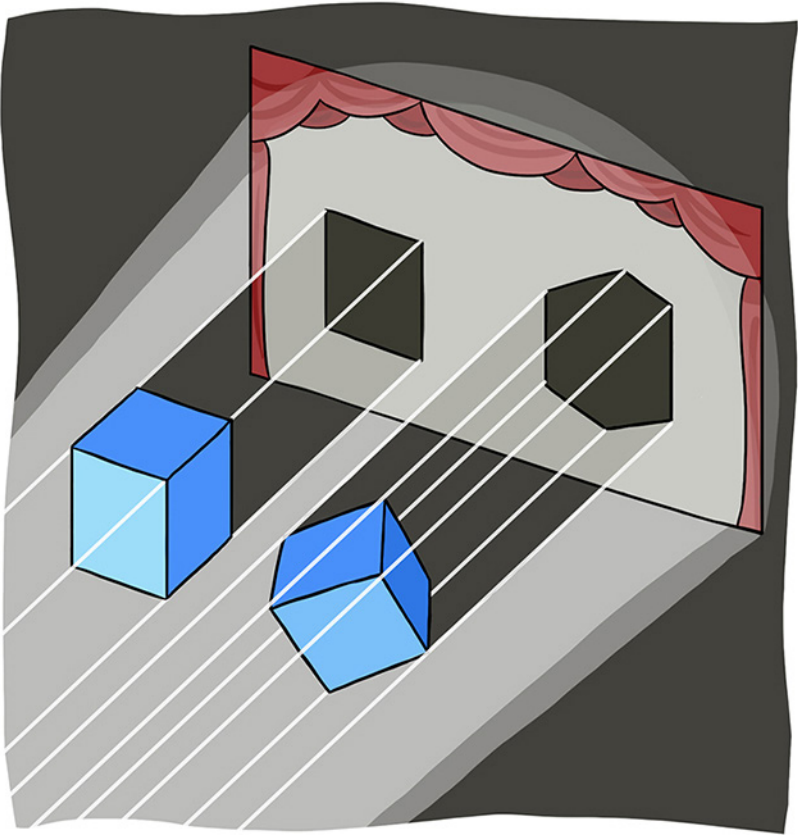
Follow me down these creaky wooden steps and be careful not to slip on the splashes of seawater. We'll walk past the cozy bunks and sneak by Lemon's corner, leading you to our very own shadow-puppet theater. It might be modest—just a simple stage in front of a few stools—but it's a gem amid the ship's confines.

Grab a stool and indulge me as your first performer. From the back of the room, I'll cast shadows in front of a torch. Tell me what you recognize. Can you decipher the rabbit, the bird, or the deer? My fingers serve as makeshift antlers!

Now, for the next one, guess the object creating the shadow. Looks like a square, right? But if you turn around, you'll see it's actually a cube that I'm holding flat against the torchlight. You might feel momentarily transported back to the 2D realm, tricked by these shadows.

My nickname for this room is "Plato's Cave." The ancient philosopher Plato crafted a thought experiment about prisoners who are chained inside a cave from childhood, their gazes fixed on a wall. Behind them, puppeteers parade objects, casting shadows onto that wall.

For the prisoners, these fleeting shadows are their only reality. They even think that they are their shadows, since that's all they see of themselves and their fellow prisoners. If the prisoners could break free and step outside, they



would see a sunlit world and realize those shadows were mere representations of a 3D reality.

Hush now, the spotlight is about to shift to our next act! Oh, yes, there's more in store than just my performance. Please direct your attention to the stage as I unveil what lies behind the curtain.

Behold the mesmerizing dance of the rotating tesseract, repeatedly turning itself inside out. This captivating form is the shadow of a rotating tesseract, much like the one we tried drawing in the previous section. What you see is not the true tesseract, but a projection here in our 3D world.

While the genuine four-dimensional shape is beyond our sight, we're fortunate to witness its three-dimensional shadow.



Plato's musings resonate deeply: Is it possible that we have something in common with those prisoners I mentioned earlier? Just as they experienced 2D shadows of a 3D world, could our everyday experience be us witnessing mere projections of a 4D existence? Is our perceived universe merely a canvas for 4D entities to cast their shadows? Plato urged us to question our perceptions, and pondering the fourth spatial dimension makes us do just that.

When I was mimicking the antlers of a deer, you might have noticed a little quirk of mine that made me particularly suited to performing puppet shows. It's a genetic gift: hypermobility. This trait allows my fingers to stretch and flex far beyond the usual range of human joints. For example, when I give a thumbs-up, my thumb doesn't just stand upright; it bends backward in an arc.

But my hypermobility doesn't stop there. With a bit more effort, if I stretch my thumb back just enough, it bends into higher dimensions. It's like watching a sticker being slowly peeled off a flat surface, my thumb vanishing bit by bit from our three-dimensional world. While I can't extend this peculiar ability to my entire body, just my fingers, it still makes for a fun party trick.

FLAT UNIVERSE

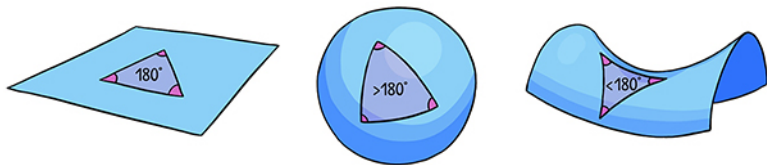
While we sit here in the theater watching the shadow of a tesseract dance in front of us, think back to our Twosville visit. We found Flatlings living on the flat surface of a patch of ocean. We examined the Flatlings from up close, but as we sailed away, we saw that their universe extended, possibly spanning all around the surface of Earth. If that's the case, then they would actually be living on the surface of a sphere and not know it.

Although it may not matter to them at all. The Flatlings will always be living in a two-dimensional space, regardless of the higher shape of that space. Their flat space may be wrapped in a ring or twisted into a pretzel in a higher dimension, but that won't change the fact that their world has two dimensions to move in and explore.

There is at least one way that a clever Flatling might be able to investigate the shape of their space. Flatlings, like us, are taught in school that, in a flat world, the angles of a triangle always add up to 180 degrees. This rule is as familiar as the idea that our sky is blue. But what if we tried to draw triangles on the surface of a balloon or upon the surface of a **Pringle**?

On a spherical space, a triangle's angles add up to more than 180 degrees. On a sphere you can draw a triangle which has three 90-degree angles, adding up to a total of 270 degrees—a concept as wild as a fish riding a bicycle! To draw this triangle you can imagine one corner at the North Pole and the other two corners on the equator, spaced a quarter of the way around the sphere from each other.

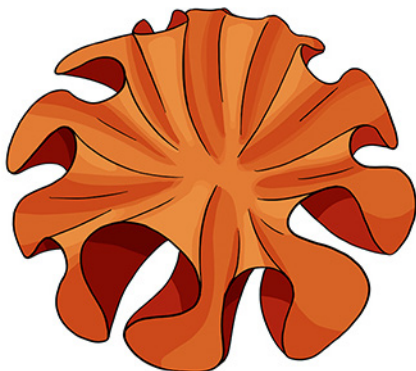
Conversely, think about Flatlings living on the surface of a tasty Pringle. It curves inward in one direction and curves outward in the other direction. On a Pringle, the angles of a triangle would total less than 180 degrees. The space is bending away, stretching the corners of the triangle outward and shrinking the angles.



A Flatling could investigate the shape of space by drawing a big triangle in their world and measuring its angles. If their world was flat, as they'd always believed, the angles would add up to 180 degrees. But if the angles totaled more than 180, their world might be curved outward, like a sphere. If less, it might be curved inward, like a Pringle.

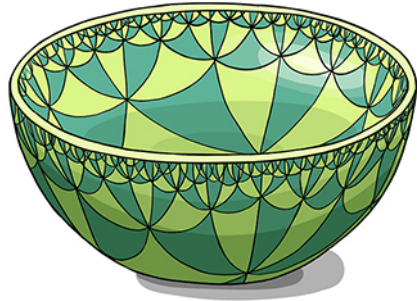
Pringles aren't the only shape where triangles have less than 180 degrees; they are just one example of a surface with **hyperbolic geometry**. A **hyperbolic surface** flares out, creating more space near the edges. Frilly hats, coral reefs, and lettuce leaves all display hyperbolic geometry. These objects cannot be laid flat on a 2D table without some part of the shape protruding out.

Imagine a Flatling city built on a hyperbolic surface, which curves away from itself at every point. It might be quite a livable city, since as you move outward from the city center, there's more and more space to spread out. A Flatling in this city might find that the shortest path between two neighboring buildings in the outer regions is actually to come back through the inner city. The shortest path between two points would be on the hyperbolic line connecting them. To a Flatling, these lines would seem straight, but to us, outside the space, they would appear curved.



It might also be easy to get lost in this hyperbolic city, for if someone looked out to the horizon, they would see more space than in a flat city. A Flatling's house, even if rather close, would appear small and faint because it takes up a smaller chunk of their field of view. If they moved away from home, their view of it would quickly shrink to a tiny speck.

No trip to a theater is complete without some popcorn, and here, I have some freshly popped for you. It is served in a hyperbowl. This bowl is decorated with a triangular pattern. To imagine a hyperbolic space, picture all these little triangles staying in their spots but scaled so they are all the same size.

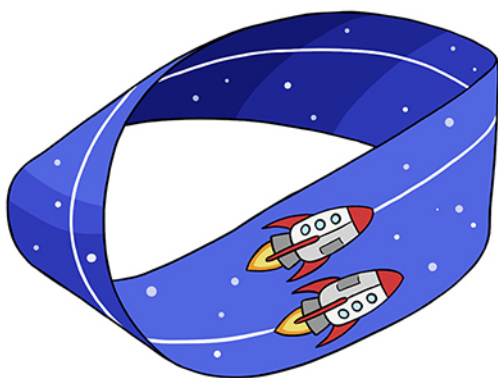


Whether it is a sphere, a Pringle, or any other strange shape, as long as it appears locally flat, the Flatlings might never notice the higher dimension of their space. Now we must wonder: If our 2D friends could be living on the surface of a 3D shape, then could we, 3D inhabitants, be living on the **hypersurface** of a 4D shape? I can't help but wonder whether we could be living upon a massive 4D hyperPringle, or even on a saddle atop the back of a hyperhorse and not know it. Could our universe be the hypersurface of a 4D hypersphere? How about a hyper-**Möbius strip**?

If our universe was twisted into a kind of 4D Möbius strip, then a rocket leaving Earth could fly endlessly in a straight line but bizarrely encounter a mirror-flipped version of Earth. Completing a second "lap" of the universe, still flying perfectly straight, it would end up back where it started, the right way up, on Earth. The light emitted in this Möbius

universe would be destined to endlessly complete loops of the strip. So, as the rocket crew looked out their window, they would see infinitely many images of themselves, half the right way up, and half mirror-flipped.

If you look out at the ocean from the top of the jungle gym on this ship, Earth itself might seem flat until you carefully watch other ships disappear over the horizon. Their lower sections disappear first, trailed last by the



tip of their masts as they move around Earth's curve, out of our line of sight. There is a spherical shape to Earth that's not immediately obvious when we focus on the ground or water beneath our feet.

So, what about the shape of the universe itself? What shape is the vacuum of space in which spherical planets like Earth and all of the stars reside? We have some of our finest minds working to uncover the secrets of our universe, constantly probing and questioning the nature of the space we inhabit. Our cosmologists have investigated the shape of our space by essentially drawing big triangles in our own universe. Not with a giant ruler and protractor, of course, but through precise measurements of the angles and distances connecting galaxy clusters that are light-years apart.

Our cosmic detectives have also added up all the mass and energy in the universe to arrive at a density, and they've used that to try and discern the shape. Mass and energy bend space as described by general relativity. Stars, galaxies,

dark matter, dark energy—everything contributes to the overall density of the universe.

The conditions for a flat universe are a bit like Goldilocks' bowl of porridge: the density has to be just right. If the universe is too dense, space could curve into a sphere. If the density is too low, it could be a Pringle.

So far, all these efforts point to our universe being flat, or at least the part of it that we can observe. Scientists have calculated the curvature of the universe to be flat within a very small degree of uncertainty. This doesn't mean there aren't curves, bends, or wrinkles caused by the massive bodies within it, but on a grand scale, space doesn't appear to have a wacky shape.

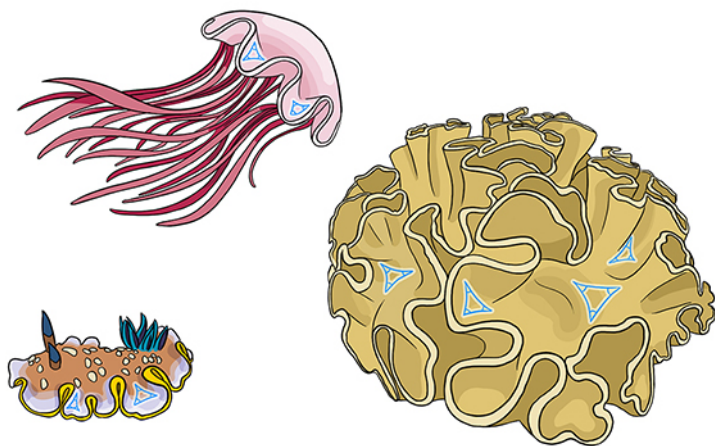
A flat universe with zero curvature could imply an infinite universe, extending in every direction, whereas positive curvature, resembling something like a sphere, might suggest that our universe is bounded to a particular size.

It's still possible that the universe is, for example, a hypersphere with a **radius** much larger than the radius of the observable universe. In that case, observing what we can see of our universe in comparison to the whole could be like someone standing in a large flat meadow contemplating the shape of the entire planet. Small triangles in hyperbolic or spherical geometry will only deviate from 180 degrees by a small amount. Within the bounds of our current knowledge, and from our spot in the meadow, the universe unfurls like a flat canvas, but we can only examine our part of it.

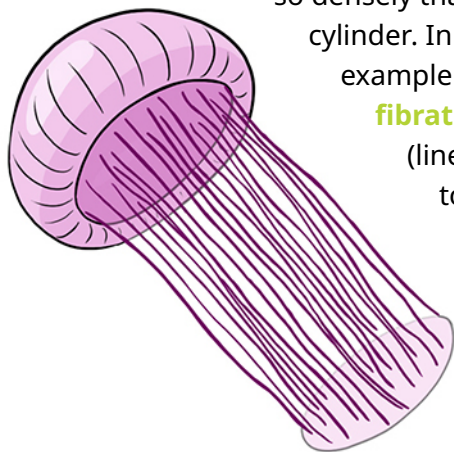
BOX JELLYFISH

If we pull back the curtains on the theater's portal windows, we see the ocean outside. Do you see the intricate folds of the coral reef? Notice how the frilly shape is similar to that on the body of a sea slug or on the circumference of a jellyfish?

These are examples of hyperbolic geometry appearing in the natural world. Perhaps such geometry manifests as a way to increase surface area, creating more contact with the water or opportunities for nutrient exchange.

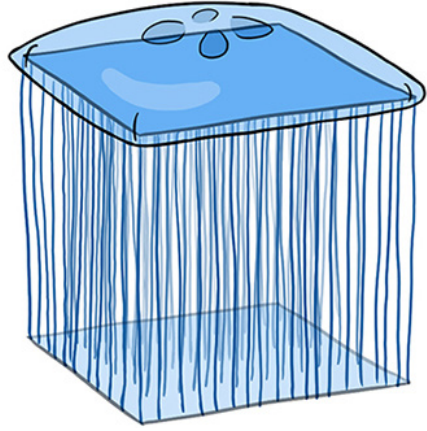


A diverse swarm of jellyfish swims past our window, a display of many shapes and sizes. There is one with a bell-shaped body and many long tentacles. Out of an approximately **disk**-shaped base, the thin tentacles grow so densely that they have created a cylinder. In mathematics, this is an example of something called a **fibration**: bundles of fibers (lines) that come together to create structures.

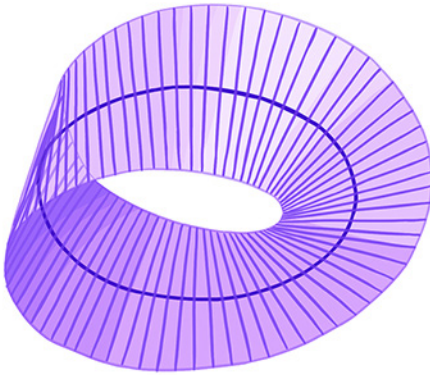


Look, there's one of the deadliest sea creatures, the box jellyfish. Out of a square base, tentacles grow densely into a box shape. In a true fibration,

there are infinitely many fibers, each infinitely thin, which completely fill the space they construct. However, the real world is always messier than the mathematical world of Threesville, so don't expect to see perfection in tentacles you encounter back home.



The jellyfishes' tentacles are not all straight. Some bundles of tentacles are twisted and wrapped, transforming 2D base shapes into 3D jellyfish. In other species, the tentacles transform 1D base shapes into 2D jellyfish, as with



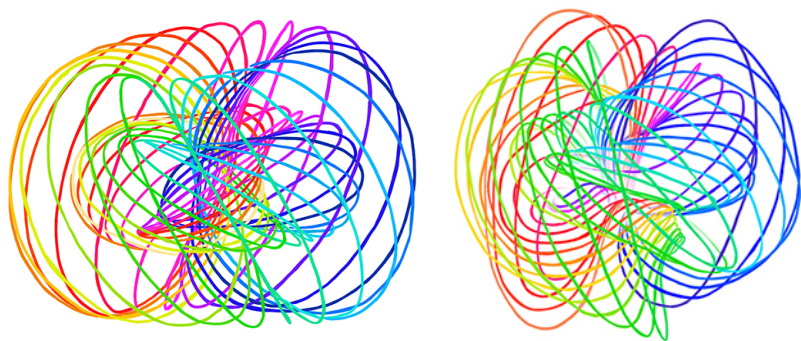
the Möbius jellyfish, formed by short, straight tentacles growing out of a circle.

The rarest and most elusive jellyfish in existence is one which swims in four-dimensional space. This highly prized species is called the Hopf Jellyfish. The lifecycle of the Hopf Jellyfish spans four dimensions. It begins its life as a **0D** egg before growing into a polyp and then eventually a sphere. In the next growth phase, every point on the surface of the sphere can sprout a circular tentacle.

Points near the top of the sphere will sprout circles with larger radii, while points closer to the bottom will sprout

smaller ones. A ring of points that share the same **latitude** on the sphere, such as those on the equator, will create circles that can come together, interlocking to form a **torus** (doughnut shape). If all the circular tentacles from one hemisphere came together in 3D space, they would form a solid torus.

I've never seen this rare Hopf Jellyfish in its entirety, but here are a few sketches I've managed to collect.



Since the Hopf Jellyfish actually lives in 4D space, these snapshots of it are merely projections into our 3D space, similar to the shadow of the rotating tesseract we saw during the puppet show. In its 4D habitat, the Hopf Jellyfish's circular tentacles form the surface of a hypersphere, which is a three-dimensional hypersurface. Every point on every circular tentacle is the same distance away from the center of the hypersphere.

Picturing what it looks like in 4D space is difficult. It is similar to asking a Flatling to imagine wrapping a circular candy wrapper (a disk) into a sphere by utilizing a mysterious third dimension. The candy wrapper becomes a sphere by pinching the circular boundary into a point without letting the interior collapse. The Hopf Jellyfish can similarly be wrapped into its hyperspherical form. Without collapsing

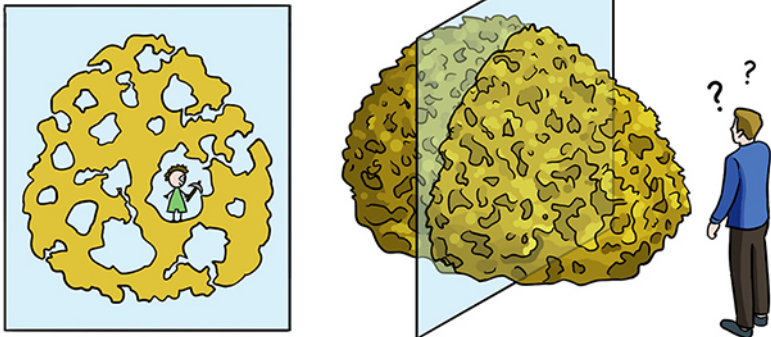
the interior, its boundary needs to be pinched together, in four dimensions, to a point.

More commonly called a **Hopf fibration**, this structure breaks the complex geometry of a hypersphere into (almost) comprehensible pieces. It provides a map from an ordinary sphere to a hypersphere. Studying it can offer insights into hyperspheres, but also the structure of **qubits** used in quantum computing, models of magnetism, and fluid mechanics.

Another curious sea creature stares us down as we drift past. Tucked away in a coral crevice is a colorful shrimp. It is hiding from the outside world, waiting for its chance to strike. The shrimp is hard to spot, but its bright tail gives it away as it scurries back further inside its hiding spot.

This shrimp makes me wonder if Flatlings back in Twosville would ever be able to “hide from us.” We could see inside their houses and even their bodies, since their entire existence is open to the third dimension. However, if a three-dimensional structure such as a porous sea sponge were placed into their world, it might offer Flatlings some privacy.

The Flatlings would only perceive the cross-section of the sponge that intersects their world. Numerous holes would appear as caves and openings. These caves might offer the Flatlings a form of



concealment because above them would be more sponge material. If a Flatling hid inside one of these holes, they'd be invisible and protected from any three-dimensional beings above the sponge.

This raises a question for us: Could there be similar "caves" in our three-dimensional world where we are hidden or shielded from a potential fourth dimension? If four-dimensional structures intersect with Threesville, what might be signs of these structures?

DUCK, DUCK, GEESE

Our inability to see creatures like the Hopf Jellyfish in their entirety is one of the limitations we face living in a three-dimensional world. Our drawings of 4D shapes are on two-dimensional paper, but even if we built a 3D model, it would still only be a partial representation of the true shape.

Another limitation that we face in Threesville is that multiple objects cannot exist in the same physical space as each other. If I sit in this chair here, you cannot also sit in the same chair unless you wish to initiate a very awkward social exchange. Objects in our world are three-dimensional and take up a volume of space, meaning that the physical interactions between objects are limited. In a four-dimensional space, our objects could share the same three-dimensional space if their positions differed in the fourth dimension.

Are you familiar with the game of Duck, Duck, Goose? It's a party game that is much more fun when played in four dimensions. In its traditional form, the game involves players sitting in a circle facing inward. One player, the "tagger," walks around the outside of the circle, tapping each seated player on the head while saying either "duck" or "goose." When the tagger says "duck," nothing happens, and they move to the next player. However, when they declare "goose,"

the tagged player must leap up and chase the tagger around the circle. The goal for the tagger is to circle back and sit in the now-vacant spot before being caught. If the tagger is caught, they must try again; if not, the new “goose” becomes the next tagger.

Now when we play Duck, Duck, Goose in the fourth dimension, the rules undergo a twist. When someone is tagged as “goose,” they chase the tagger as usual. However, due to the fourth dimension’s unique properties, multiple players can occupy what seems to be the same chair in our three-dimensional space. This is because, in the fourth dimension, they are actually positioned at different points along the unseen fourth axis.

So, while chasing and finding a chair, players must consider both the traditional 3D space and the elusive 4D space. This adds an intriguing layer of complexity to the game, because there might be hidden players, which, when tagged give rise to a whole team of “geese,” making it much more difficult to avoid being caught.



HINTON'S CUBES

Pardon my urgency, but I've sighted a vessel straight ahead at 12 o'clock. Can you discern that? On the horizon, a dark, cube-like silhouette steadily grows more prominent as it nears us. Pirates, to be exact. Please, everyone, remain calm. Should they board our ship, it's our safe they'll target, not the passengers. Lemon, stand by the cannons!

Dear friend, would you like a turn with my binoculars? Now that the pirate ship looms closer, details emerge on its side. If you squint, you can see that the ship's shimmering facade boasts a beautifully engraved name: *The Hinton*. We must act swiftly now. Those aren't mere pirates; they are Hinton's cubes, a notorious legion of deceptive blocks.

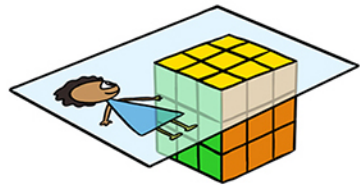
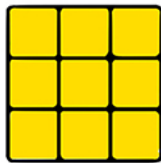
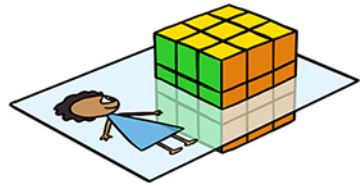
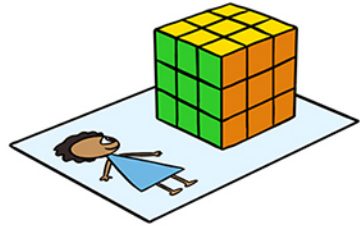
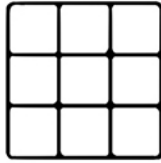
Do you see that their boat has a peculiar, multi-layered structure, moving and shifting as if its very essence were fluctuating between our dimension and another? That's no mast and sail; on board are 81 cubes. When they move about, they give the impression that the ship itself is warping and twisting. A geometric construct used to intimidate enemies into thinking they are about to encounter a 4D monster when it's only a bunch of measly 3D cubes, it's like a red butterfly using mimicry to appear poisonous when it really isn't.

Hinton's cubes are clever—I will give them that. Remember a moment ago when we drew a cube on paper, representing a three-dimensional object in two dimensions? Hinton's cubes are a way to visualize a 3D representation of a 4D object.

By spending time with Hinton's cubes, one might gain a more intuitive understanding of the properties and behaviors of objects in four-dimensional space. However, genuinely grasping the fourth dimension remains difficult, and one would not be wise to spend too much time with this particular bunch.

Most of us are familiar with the classic Rubik's Cube, a three-dimensional puzzle made up of 27 smaller cubes, with each edge comprising three cubes. Each face of the cube sports one of six colors. The center cube on each face displays a single color, edge pieces display two colors, and corner pieces display three.

Now, imagine presenting a solved Rubik's cube (with its colors unscrambled) to inhabitants of a two-dimensional world. They would only observe one nine-square slice at a time. However, by observing the changing colors as the slices moved through, they could start to piece together a mental image of the full three-dimensional cube.



Hinton's cubes could help us imagine a Rubik's cube from the fourth dimension. Since a 3D Rubik's cube has three mini-cubes along each edge, to extend this into four dimensions, we'd replicate the entire 3D cube three times along this new, fourth axis. This results in a staggering total of 81 mini-cubes (27 cubes from the original, multiplied by 3). As this four-dimensional tesseract intersects with our three-dimensional

space, we would see cross-sections of 27 mini-cubes at a time—much like a Flatling would see slices of nine squares.

In the drawing below we see one side of a tesseract, followed by the middle cross section, then the other side, as if it is passing through our 3D space.



Much like the Flatling observing slices of the 3D cube, we could witness the changing colors of these cross-sections and attempt to infer the higher-dimensional structure. To us, each cross-section might appear as a scramble of colors for each set of corners, edges, and faces. Yet, with time and observation, we could begin to understand the mathematical relationships binding these colors, providing a glimpse into the complex beauty of a four-dimensional puzzle.

To learn more about the cubes and to attempt mentally rearranging and disassembling them, you can read the teachings of Charles Hinton, who invented the cubes in his 1888 book, *A New Era of Thought*.

Watch out; here they come! The cubes are pulling up now, dismantling their illusion and boarding our ship.

"Begone, Hintons!"

"Silence, foe; I am the new captain of this ship. Call me Captain Cube. These are my compatriots, Eyepatch McTwelveEdge, Hypercube Hook, Peg-Leg Polygon, 6face ..."

Dear friend, I shall spare you the gory details of our heated exchange and the names of all 81 pirate cubes. While you

were wisely hiding in your quarters, Captain Cube demanded to know the location of our onboard safe. Given the pedigree of my guests, such as yourself, I decided not to put up a fight. I walked with defeat toward our onboard safe, which contains many treasures, including my gems and the very fare you paid me for this tour.

I was ordered to open it, so I reluctantly agreed. As I twisted the dial, entering the complex sequence of the combination, I could feel every eye on me, watching, waiting. Each click was amplified by the tense silence. The door of the safe swung open, revealing its contents. The sharp gasps from the pirates echoed my own surprise. We all stared at the void in disbelief. The safe was ... empty.

Captain Cube screeched with betrayal. Hypercube Hook stuck his curved hook to my neck and demanded to know where we kept our treasures if not in the safe.

"I ... I don't understand," I muttered, my mind racing. I had checked the safe before we set sail. I had seen it with my own eyes—my savings had been in there, nested securely within. But now, it was as if they had vanished into thin air. The lock hadn't been broken.

Eyepatch McTwelveEdge felt around inside the safe, examining it for any false bottoms. He emerged holding a folded-up scrap of paper. It was a note written in a beautiful, flowing script. It read, The fourth dimension sends its regards.

There was a moment of shocked silence before it all sank in. My gems hadn't been stolen by other pirates or hidden by me. They had been taken by something or someone from the fourth dimension. The pirates' heist had been thwarted by an entity beyond our understanding: a creature of higher dimensions.

As if on cue, Lemon pranced onto the deck. She brushed past Hypercube Hook, her tail high in the air, completely

unfazed by the intense drama unfolding around her. Her carefree demeanor brought a small smile to my face and seemed to defuse the tension on deck.

“Well then, if that’s settled,” I said, grabbing the note from McTwelveEdge and folding it into my pocket, “I believe it’s time you took your leave. The fourth dimension has already had its say.”

With a flurry of grumbles and protests, the geometric pirates shuffled back to their vessel. Captain Cube cast a final, lingering glance back at our ship as he retreated. They sailed away, a glimmering fake rainbow tesseract, leaving us in peace, their dreams of riches dashed by an entity they could barely comprehend and wished they resembled.

Once again, our ship is ours, and we are alone in the expanse of the sea. My onward motivations have now changed: I do not simply want to visit the fourth dimension to show you around; I must go there to retrieve my stolen gems.

PERFECT

There's no such thing as a cube,
Even if you think you saw one on YouTube.

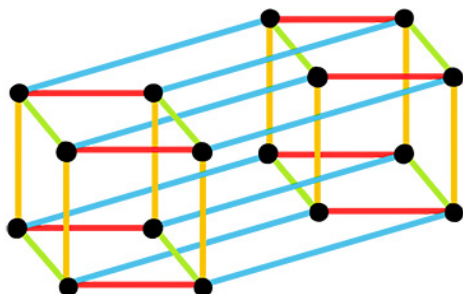
There's no such thing as a sphere,
At least not in our messy world here.

There's no such thing as a triangle,
They're concepts for our minds to wrangle.

All you've seen are approximations,
Zoom in, you'll see the fluctuations.

Real shapes are imaginary,
They exist in the airy-fairy.

And isn't it sad that we'll never see one?
Only in our thoughts do we get that fun.



PART 4:

FOURSVILLE

My heart is pounding in my chest with anticipation. The moment we've dreamed of and prepared for is finally here. Alright, friends: Brace yourselves. Do you feel a strange sensation of vertigo washing over you? I'm taking you to a space where you will have more freedom than you've ever dreamed of. I'm taking you to **Foursville**.

To get there, we turn the boat and everything inside it by a quarter of a full turn. That's the usual angle of separation between two sides of a square when they meet at a corner. You may wonder what we are measuring our quarter turn rotation with respect to—well, we are turning with respect to ourselves! We are turning to face a new path that is perpendicular to all of our previously known three directions. Our new track meets north-south, east-west, and top-bottom at a 90-degree angle.

As we rotate to face the new dimension, the familiar sights of our three-dimensional world warp and twist, becoming shapes that are too complex for language to capture. Our collective gasps resonate through the cabin as we see an alien spectacle of floating blobs and shifting, disconnected corners.



Remember those knots that were tied in the rope on the railings? Well, take a look at them now, and you will see them slipping untied. A knot that is strong in three dimensions is no match for a **fourth dimension**. The knots untie themselves with the ease of a snake slithering out of its coiled rest spot.

We're here, my friends: Welcome to Foursville.

THE HYPERPEACH TREE

Our first marvel is just ahead, towering over us in all its grandeur: a peach tree of four dimensions. If we keep our heads still, it almost looks like a normal tree. As soon as we move our heads, though, the shift in our perspective reveals a boundless network of interlocking branches. This pattern of branches and leaves is symmetrical in ways we've never seen in a tree back home.

The hyperpeach tree is interesting in that its life cycle spans four spatial dimensions. The tree starts as a small seed, a zero-dimensional point. The instructions for life are locked up inside, unable to move until the seed encounters the nutrients in the soil that it needs to start growing.

From the seed emerges a thin stalk, growing toward the light. The stalk of a plant can be said to be approximately one-dimensional. The water and nutrients inside flow along a line. A bug crawling up the thin stalk could crawl up or down but not meaningfully from side to side.

From this stalk, leaves shoot out. These flat round surfaces collect light to turn into energy for the plant. Each leaf is approximately two-dimensional. A bug can crawl around the surface of the leaf, restricted to the flat expanse, unless it has wings to fly away.

From the end of a branch, a bud may form and grow into a fruit. Fruits, as we know them, are three-dimensional. A normal peach has a fuzzy surface and a juicy inside. Taking

a bite pierces all three dimensions. But then there's the hyperpeach—the four-dimensional extension.

Hyperpeaches aren't your usual garden-variety peaches. These peaches, my friend, are extraordinary. From the branches hang all manner of differently sized peaches, from tiny to giant. As the wind rustles through the tree, the peaches grow and shrink in size. Some of them appear to be suspended in mid-air—a pink delicacy floating amid the leaves—but as the branches rustle and the size grows, a stem will pop momentarily into view.



Picking one of these peaches isn't as simple as reaching out and grabbing it. No, you must navigate around the hyperspherical fruit, deciding on the right angle from which to pluck it from its branch. But the taste—*oh* the taste, my friend. The juices seem to come from inside your tongue.

Why are the hyperpeaches so odd-looking? To us, they are hard to look at because our vision is unfortunately still limited by our anatomy. Our eyes still see **2D** images of this world and construct a **3D** reality, much like back home. The light-collecting retina inside our eyes is a flat surface.

A **4D** creature has spherical light-collecting retinas, and they see an entire 3D object at once. Walking around a hyperpeach, a 4D creature's eye stitches 3D **cross-sections**

together seamlessly to perceive in 4D. Our brains are working overtime to cobble together the confusing signals we are now feeding it.

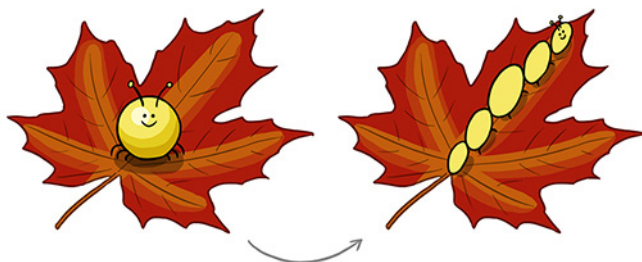
Much of constructing what an object looks like comes from prior experience. We know that a soccer ball will be round at the back without looking to check every time we see one. With no experience of a hyperpeach, our brains don't know how to fill in the blanks of the picture until we see it.

Look closer, near the trunk of the hyperpeach tree. Climbing along a branch, you'll notice a peculiar green caterpillar with a body consisting of **spheres** of various sizes. This intriguing creature resembles a string of 3D pearls.

It uses its unique arrangement as camouflage. By assuming a three-dimensional form, it is hiding its true four-dimensional self. A predator—maybe a hyperbird looking for a juicy meal in the ana-kata direction—might not even notice this caterpillar, or mistake it for a flat, nutritionless meal.

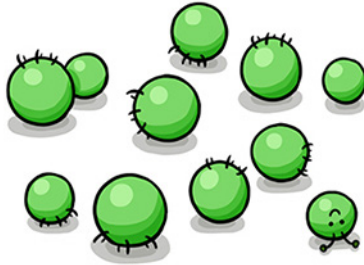


Imagine a round bug back home had entered a cocoon and safely emerged as a thinly sliced version of itself, with the slices laid flat in a line. The bug would still exist, but to a bird flying overhead, it would disappear into flatness.



Allow me to venture an introduction to our caterpillar friend. I'll gingerly guide the ship closer. It's difficult not to brush against the hidden branches of the hyperpeach tree as I do so. This brushing motion sends vibrations throughout the tree, knocking several hyperpeaches to the ground. Oh, peaches! The sudden noise caught our friend off guard, startling it into a frenzied reaction.

With a sudden "pop," the caterpillar bursts into an array of tiny spheres, scattering across the floor like an impromptu game of marbles. What an unexpected spectacle!



As we watch, the spheres start to frantically reconvene. Their rapid movements are mesmerizing. One by one, they disappear from sight, vanishing as though swallowed by the invisible jaws of our vision's edge. Just one single green sphere remains in our vision.

The other spheres have reintegrated into the bug's true form: a four-dimensional **hypersphere**. Back home, we would have to be content with losing sight of the hyperbug, but copy my movements: All we need to do is rotate our head slightly in the ana-kata direction and ... there! We can watch the bug as it crawls away.

Foursville introduces an additional axis perpendicular to all others we are familiar with. It provides the hyperbug with height, width, depth, and an "extra something" beyond our normal perceptions.

DR. WUMBO

Our attempt to befriend the caterpillar didn't go so well, but I'm going to take you to meet a friendlier 4D being. This is the home of Dr. Wumbo. I know him because he is a medical doctor who offers a special service: medical examinations of lower-dimensional beings. I've been coming to him for any ailments for several years now. I had a slipped disc in my back that he was able to fit back into place without surgery. (Look, no scars!) He reached into my spine through the **fourth dimension**, so didn't pierce any skin.

Behold, there's Dr. Wumbo! Glimpses of his form appearing in disjointed segments hint at his complex anatomy. Each movement through the ana-kata axis reveals more of his friendly face and octo-limbed structure. With four arms and four legs, he reminds me of the ocean's most graceful dwellers.



Try not to think about the fact that Dr. Wumbo can see inside of your body. (Oops, sorry to have reminded you of that just now!) I can see Dr. Wumbo looking you over and under. His diagnostic gaze is as comprehensive as a CAT scan, without the exorbitant fee. A CAT scan back home takes many X-ray images and stitches them together to form a 3D image. If you want to imagine having vision like

Dr. Wumbo, just watch an animated CAT scan. There are some nice ones online that show the insides of fruit.

If a 2D Flatling had any hope of visiting Threesville for medical attention, they would need to imbue themselves with at least the slightest thickness in the third dimension.



Being paper-thin is okay, since paper exists just fine in Threesville, but being infinitely thin would be a problem. No Threesville doctor would be able to interact with that being.

On our current excursion, we likewise will find we have the slightest thickness in a fourth dimension. To any 4D native, we would still appear relatively flat, but this thickness allows us to continue our tour here. It also provides us with a kind of backing, shielding a portion of our bodies from Dr. Wumbo's gaze. Picture again the paper-thin Flatling who stumbled into a Threesville hospital; when viewed from the front, you could peer directly inside its body, but from behind, all you'd see was the bare white thickness of the paper.

Our visit, while enlightening, isn't merely a courtesy call. A special passenger is napping in the medical quarters on our vessel. I will go and fetch her now. I'm sure she will be very pleased to know we have arrived.

"Dr. Wumbo, my friend, I present Mary and her devoted partner. They are soon-to-be parents!"

Dr. Wumbo nods, his penetrating gaze coming to rest on the nervous couple. "Welcome. Will you please both follow me to the surgery quarters?" His deep voice echoes. "All is prepared for a peaceful and effortless arrival. There is nothing to worry about."

Mary exchanges a glance with her partner before they follow Dr. Wumbo toward an arched door, shimmering slightly as if caught between dimensions. Let's step back and give them privacy as they disappear through the doorway.

For Mary, nine months pregnant, this is a rare opportunity. She will have a painless delivery of her baby. Dr. Wumbo will extract the baby with ease, as there are no organs or skin in the way of its 4D exit. The baby will take its first breath just beyond the parents' view and then be handed back to them.

Mary is lucky to have the financial means for this trip and the courage to explore the option. The medical tourism component of my voyages is how I manage to fund my lifestyle, but it operates a bit like a black market. Regulations back home haven't quite caught up to the opportunities present here.



"Oh, here comes Dr. Wumbo, two proud parents, and their new baby!"

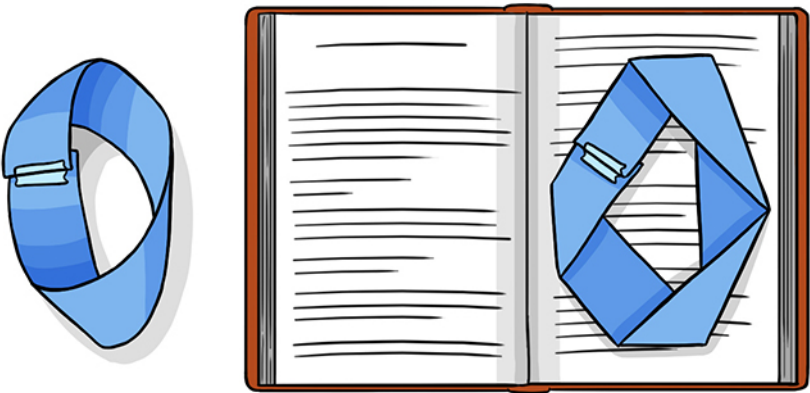
A well-meaning family member has packed a present for the new baby in Mary's travel bag. It's a Klein bottle, and if you want to see a spectacle, then watch silently with me as Mary's partner fetches it from the bag.

For a second, think back to the humble **Möbius strip**, a two-dimensional surface with a half twist. A Möbius

strip is just a mathematical idea; it doesn't exist in the real world because it has no thickness. However, you can make something similar to a Möbius strip by putting a half twist into a ribbon and gluing the ends together.

The result is a shape with only one side. Trace a path around the strip and you will end up where you started. Can you imagine trying to fill the volume contained by the Möbius strip with, say, milk? Of course not, because there is nothing to fill! There is no inside or outside of the Möbius strip.

However, that's only the case for a Möbius strip embedded in a world of three or more dimensions. If we placed our Möbius strip into a heavy book and slammed the pages shut, the strip would be flattened and folded over itself. Now occupying just two dimensions, this former Möbius strip, which now has only straight edges, divides the area of the page into two distinct regions, an inside and an outside.

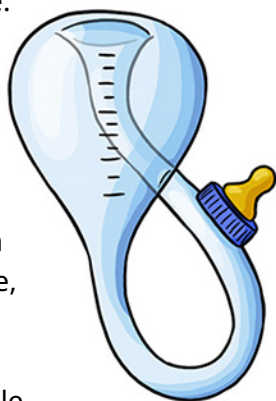


A Flatling from **Twosville** could be trapped within a flattened **Möbius strip**. From their perspective, they would see only the sides facing inward, unaware of the strip's unique properties. However, with our knowledge of higher dimensions, we know that by unflattening the shape and

restoring it to its 3D form without intersections, the barrier disappears—it's no prison at all.

A Klein bottle shares similarities with the Möbius strip. Here in Foursville, the bottle forms an endless loop, free from any self-intersection—something that's difficult for us to imagine. Any glass models of Klein bottles back home in Threesville have been 'flattened' to 3D. The illustration I've included below has been flattened even further to this 2D page. In these attempts to depict a Klein bottle with any less than four dimensions, a self-intersection is unavoidable.

It's understandable that someone would gift a Klein bottle to a baby being born in Foursville. After all, the family member might have heard that it's a classic four-dimensional shape. The "bottle" part makes it seem like it could hold liquid, and indeed, the self-intersecting 3D models can. Unfortunately, as we watch Mary's partner pour milk onto the bottle, we see the problem. A real Klein bottle only has one side. There is no inside or outside of a Klein bottle here in Foursville.



As milk splashes on the floor, I am placing a **tesseract**-shaped bucket down to catch the overflow. The top cubic face of the bucket has been removed to allow liquids to enter and remain contained.

4D PLAYTHINGS

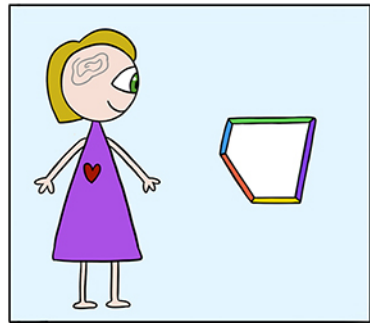
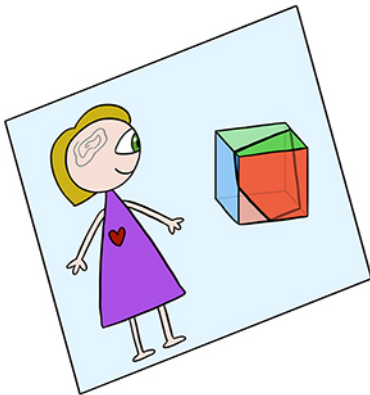
Dr. Wumbo, ever the thoughtful host, has gifted the baby a trove of four-dimensional playthings, cherished mementos of its first moments. Each baby born here leaves with a set of these toys—real collector's items. Let's take a look through the contents.

There is a box of dice. The tesseract-shaped die has 24 square faces, but a 4D creature would barely be able to see these flat squares, so they are not the numbered part. That would be like us labeling the edges of a die back home. Instead, the 4D creatures label each of the cubic cells of the tesseract. There is one solid 3D cube facing each end of the four dimensions, making eight in total.

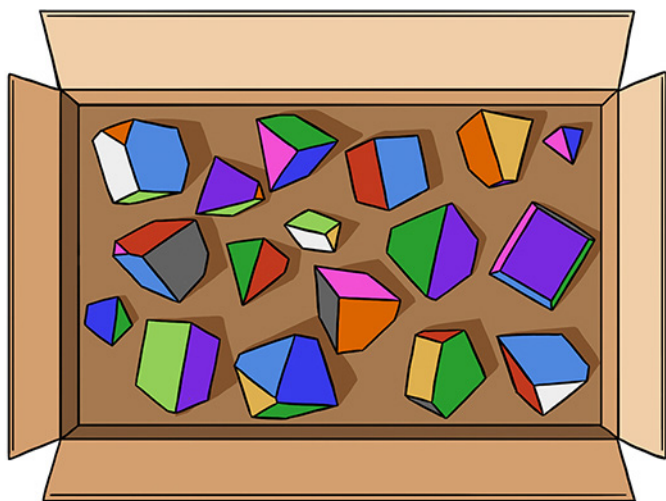
Each of the eight cubic sides is colored differently. There's a red, blue, green, yellow, pink, orange, purple, and white side. They correspond to the numbers 1 through 8. The 4D gamers roll dice onto a 3D tabletop in much the same way we roll dice onto a 2D tabletop. On a 3D die, numbers on opposite sides will add up to 7, but on a 4D die, numbers on opposite sides will add up to 9.

With our vision, we will see a 3D cross-section of the tesseract resulting from each dice roll. The colored side that faces up will be our result.

As an analogy, consider a Flatling who managed to roll a 3D die, with each of its six faces painted a different color. From their perspective, they would only see a flat cross-section on each roll. The color of the top line in that cross-section would be their result.



Dr. Wumbo has been generous enough to give an entire box of dice in case a few get lost. If you shake the box, the dice will come to rest on a variety of different angles, each with a different cross-section visible to our eyes. This makes them look like a bunch of strangely cut candies instead of their true tesseract form. Shake them again, and they will jostle around before coming to rest in different positions, exposing a new set of sides as if you had exchanged them for a new box of dice.

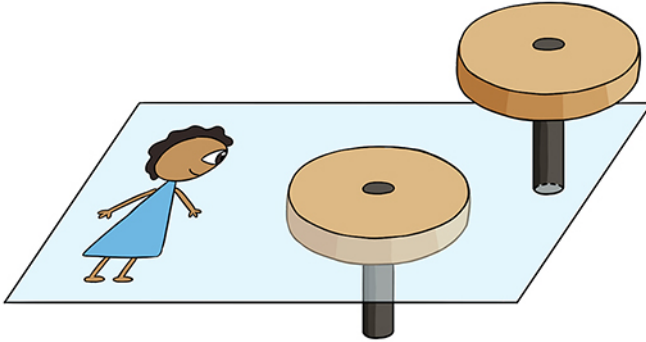


The gift bag also contains a 4D toy car. At first glance, the car might not resemble anything you're familiar with. It's only when it moves and you examine it closely that its car-like qualities become apparent.

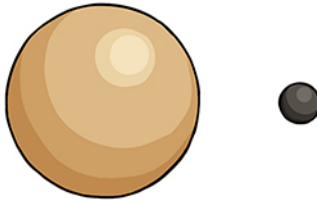
Staring into our gift bag, we can see eight solid gray spheres, which are cross-sections of each wheel. If you jostle it a bit, they turn into eight small black beads; these are the axles that are situated inside the wheels. They connect to the rest of the car which comes into view as a confusing mess of pipes, wires, and car parts. The toy steering wheel is a

sphere, of course, as a simple circle wouldn't offer enough control to navigate a road in Foursville.

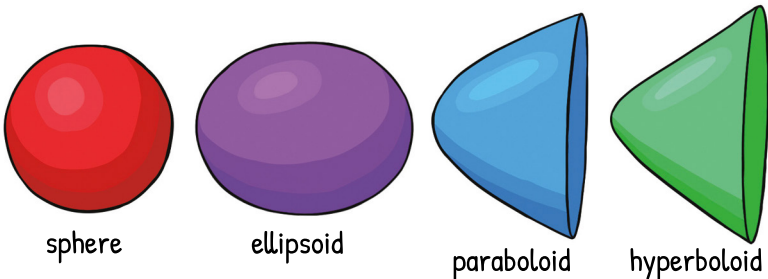
3D wheel passing through 2D

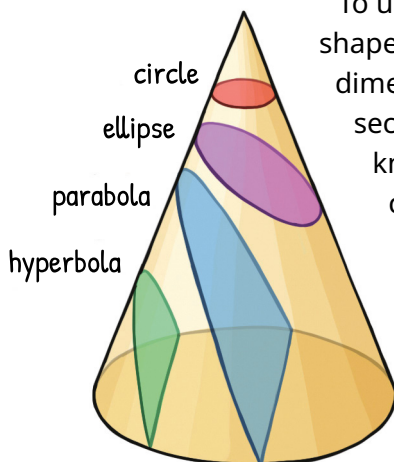


4D wheel passing through 3D



Be careful not to brush your hand against the pointy end of another toy, the 4D cone. In your palm, it begins as a perfect sphere, but with a mere twist along an unseen axis, it changes, morphing into strange solid shapes. These shapes are the **ellipsoid**, **paraboloid**, and **hyperboloid**.



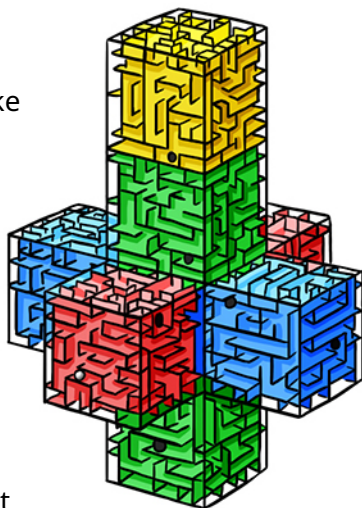


To understand why we see these shapes, think about a three-dimensional cone and its cross-sections. If we slice a sharp knife through an ice-cream cone, the exposed face would either be a circle, **ellipse**, **parabola**, or **hyperbola**, depending on the angle of the slice.

When the 4D cone is visible to our human eyes, we see a slice

of the greater cone, so we are seeing the conic sections mentioned above, just elevated up one dimension.

Within this gift bag also lie wonders like the **cubinder** (a cube stretched along the fourth dimension to make something like a cylinder) and **spherinder** (a sphere stretched along the fourth dimension), and an assortment of 4D **tori**. Then there's the puzzle maze, a handheld challenge that requires a rotation along the ana direction to allow a metal ball to jump over barriers. Back in the third dimension, kids might throw this toy out, thinking that it's impossible.



As our visit nears its end, a practical matter arises. "Dr. Wumbo," I inquire, "have you perchance heard any

rumors of gem thefts in these parts? Our vessel has been victim to such a theft, much to our dismay.”

Dr. Wumbo responds in his usual polite, cryptic way, “My dear captain, no such gems have crossed my gaze. You might inquire at the laundromat yonder, where rumors find solace. But, oftentimes, the treasures we seek can be found by searching inside one’s own heart.”

With a nod and a newfound clue, we prepare to venture toward the laundromat, the next stop in our dimensional odyssey.

THE LAUNDROMAT

To the unsuspecting eye, the laundromat is just another neighborhood spot for cleaning clothes, but in reality it’s a front for organized crime. It is built around a three-dimensional laundromat and allows for the silent pilfering of unmatched socks, which are highly valued in Foursville.

Here, the washing machines possess a covert connection to their counterparts in our familiar three-dimensional world through concealed 4D pipes. People have long suspected their socks are disappearing, but very few have managed to guess correctly that this is where they have ended up.

Let us steal a glimpse through the glass windows. I must confess, a wave of trepidation washes over me at the thought of stepping foot into such an unscrupulous place. It’s a potential den for all manner of otherworldly felons. Yet, my eyes can’t help but be drawn to an immense mound of socks—a mountain that touches the ceiling! I suspect that they are concealing my gems within the socks.

Now it’s time for us open the door and cautiously creep inside. Your role is to concoct a clever ruse while I probe the sock mountain. The tiled floor’s checkered pattern, reminiscent of a **spacetime** grid, extends beneath our feet.

A quick rummage through the superficial layers of the heap yields no glimmer of jewels. And then—disaster strikes! With a startled cry, I tumble into an unseen pit, an insidious trap laid in the invisible kata dimension, hidden from my three-dimensional sight.

Please, my favorite passenger, can you throw me a line? Tie some of the socks together. There are several scattered on the floor near you. Throw it to me, kata!

Oh, thank you! I've got it. I'm out, but look behind you! I see the slivered forms of the laundry thieves coming. Let's run!

Running is difficult, as when we are trapped in a nightmare. The tiled floor is not flat in the third dimension, and thus, as we run, we dip in and out of the kata direction, making only slow progress back toward the ship.

A fellow passenger appears at the rails of the ship holding a bright orange life ring. They can see we are in trouble. The life ring is part of a more complex four-dimensional shape called a torisphere, which is a three-dimensional ring (torus) at every slice along the fourth dimension. This life ring is attached to the ship with a rope. We catch the life ring and pull ourselves in with the rope. It prevents us from slipping too far away from the ship in any direction, including kata.

Phew! We're back onboard, with the anchor pulled up in record time. The laundry thieves are still chasing us. Surely they can see that we have retreated. Their continued aggression makes me wonder if they are also victims of theft. It seems like no matter what dimension we are from, we are all trying to recover something stolen from us.

MATHEMATICS OF THE FOURTH DIMENSION

Despite this being a mathematical cruise, we haven't encountered much of what people commonly think of as "mathematics." I'm referring to numbers and equations.

I want to remind everyone that math isn't just about numbers; it is more about relationships. Math describes the relationship between numbers but also the relationships between shapes, spaces, and concepts.

Sometimes, we struggle to visualize these relationships. Consider, for example, the relationship between four perpendicular spaces and how they might form a fourth spatial dimension. Where our imagination fails us, equations can step up and describe this relationship in a meaningful way.

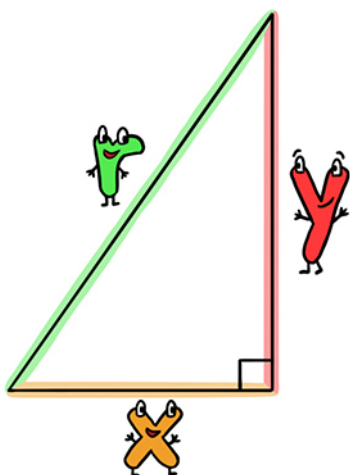
Equations are powerful: Changing a number on paper from a 3 to a 4 is as simple as erasing the number and scribbling a new one in its place, but the effect is vast. A single pencil stroke can catapult us through dimensions at speeds my ship could only dream of.

To see what I mean, let's look at some examples of relationships and see how powerful they are. For those who might find themselves in unfamiliar waters when it comes to the language of mathematics, fret not: I'll be your gentle guide.

Let's acquaint ourselves with some fundamental characters of our story: x and y . These are dear friends of mathematics and belong to a gang called **variables** because we can vary their value. While they might sound abstract, they merely represent any number you can think of.

These friends of ours can perform tricks, such as multiplying by themselves. This is called **squaring**. For example, " x squared" is denoted by x^2 , which means x multiplied by x .

Our variable friends can come together to represent the sides of a triangle. Pythagoras, a thinker from ancient times, recognized a relationship between sides of a right-angled triangle. That's a triangle with one angle that's a quarter of



a full turn. On this triangle, the two shorter sides can be represented by x and y , while the longest side (called the **hypotenuse**) is represented by another variable. We can call them r .

These particular names for our variables are merely a convention. If you don't like the sound of them, you can swap them out for anything you want. You

could label the sides of the triangle with your three favorite emojis, if you prefer. I'll stick with x , y , and r , which might help make this explanation feel more familiar and, hence, easy to grasp.

Pythagoras proved that the square of the longest side, r^2 , is found by adding together the squares of the other two sides. In the language of our variables, this translates to $x^2 + y^2 = r^2$. This relationship is always true as long as our triangle has two sides that meet at a right angle.

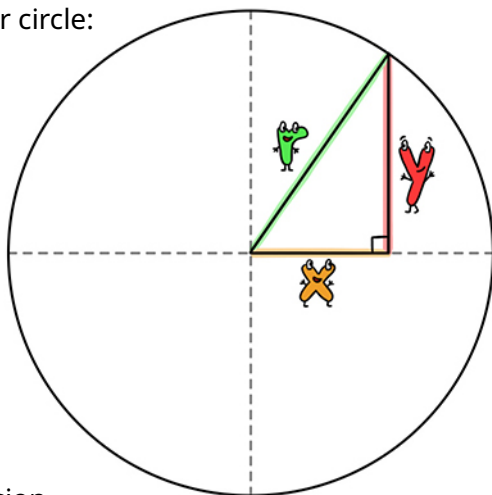
If we were to line up all of a variable's possible values in a neat and orderly line, we'd have an axis. The x and y axes extend from negative infinity to positive infinity and cross paths in the middle, at the value *zero*.

Let's imagine a circle centered on this meeting point. Every point on the circle's edge is a fixed distance away from the center. This distance is called the **radius**. When we choose a point on the circle's edge, the horizontal distance to that point from the center is x , and the vertical distance is y . A right-angle triangle forms from these two distances and our circle's radius is the hypotenuse. A familiar relationship

appears to describe our circle:

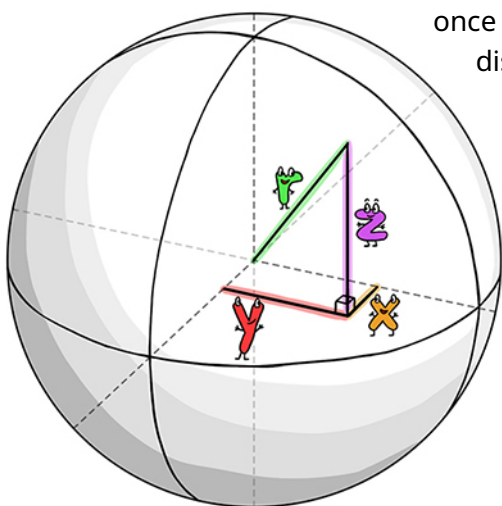
$$x^2 + y^2 = r^2$$

No matter where on the circle you choose to stand, this relationship between x , y , and r holds true. It's the circle's silent promise, binding all its points to its center.



Let's go up a dimension to three-dimensional space. Our old friends will follow us, but we also get to meet another variable, z . This newcomer represents depth, stretching away from and toward us.

A circle is a two-dimensional ring of points, whereas a sphere is a three-dimensional shell of points suspended around its center. Every point on the sphere is the same distance from its center, and, once again, we call this distance the radius, r .



In 3D space, our equation extends to accommodate the newcomer, z . With our knowledge of the circle, the sphere's equation naturally follows:

$$x^2 + y^2 + z^2 = r^2$$

The values we use for x , y , and z are the distances that we have traveled in each direction to a point on the sphere. This equation binds the entire ensemble of the sphere to its heart, ensuring that all the points dance at arm's length—no more, no less.

Now, if you're seated comfortably, let's push the math to four dimensions. Here, we introduce another new variable: Let's call them w . If x , y , and z give us width, height, and depth, respectively, w is a direction wholly alien to our typical spatial senses. It is the direction we have so far called *ana-kata* (positive w -negative w).

Mathematicians often use w here because it's close to the other letters in the alphabet. Still, you could have picked any letter, symbol, or emoji to be the name of your fourth dimension.



In this 4D space, the sphere blooms into a *hypersphere*, the name given to a sphere in any higher dimension. The specific name for a 4D hypersphere is a **glome**. Every point on this glome is equidistant from its center, like our circle and 3D sphere. Only now, this distance spans four dimensions. The equation of this 4D sphere? It's surprisingly simple, given the complexity of what it describes:

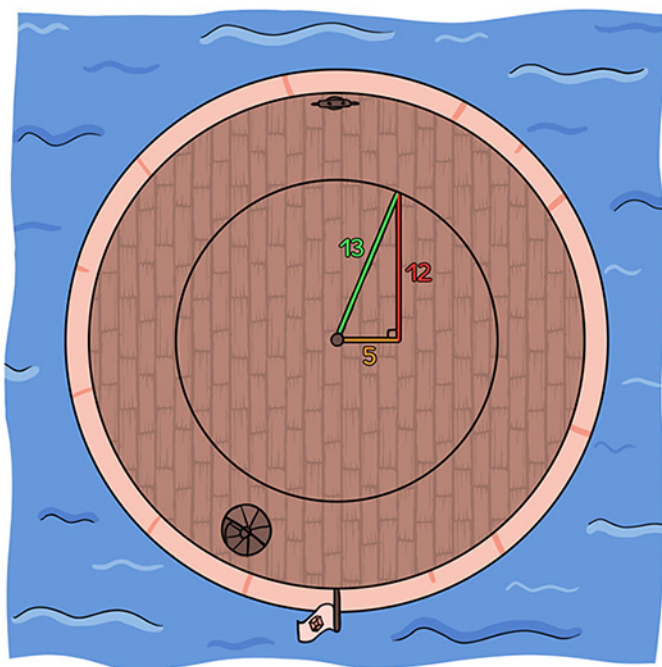
$$x^2 + y^2 + z^2 + w^2 = r^2$$

A note of caution: The word *sphere*, when used in a casual context, often refers to any ball-shaped object, but in the precise language of mathematics, a sphere is only

the shell of points on the surface. If we want to include all the points inside this shape, we call it a ball, and these, too, can be extended to higher-dimensional **hyperballs**. The 4D hyperball is called a **gongol**.

The life ring saved us mere moments ago in the laundromat because it was attached to a rope that we used to pull ourselves back to the boat. That length of rope can be used to trace out spheres of all dimensions. Using it, combined with the equations above, we can declare a sphere of safety around the boat. The life ring couldn't rescue us if we ventured outside of the **safety sphere**, meaning outside the range of the rope.

One end of the rope is fixed to a sturdy hook on the deck, this is the center of our safety sphere. I'll give you the other end to hold. I have unwound some rope equal to 13 of your steps in length and I want you to walk around the deck,



pulling the rope tight and using paint to mark out your path. The deck is a two-dimensional space, since you can walk in two distinct directions on it. You can walk left-right, our x coordinate, or forward-backward, our y coordinate.

As you walk, you will trace a circle that is described by the relationship $x^2 + y^2 = 13^2$.

Here x and y are the number of steps you took from the center in each direction. Any diagonal steps are a combination of each direction.

For instance, if you took 5 steps toward the starboard side of the boat (x), then you can take another 12 steps toward the bow (y) to maintain tension in the rope: $5^2 + 12^2 = 13^2$. As you walk through all possible combinations of x and y that satisfy this equation, you trace out the circumference of a circle.

In three dimensions, we add the possibility of moving upward, which we'll represent with the z coordinate. The safety sphere that you will trace by holding the same length of rope tight will now be described by the relationship $x^2 + y^2 + z^2 = 13^2$.

If you move 3 steps to the starboard side (x) and 4 steps toward the bow (y), then you can climb 12 steps up the rope ladder (z) and the rope will stretch perfectly to its full length of 13 steps. We know this since $3^2 + 4^2 + 12^2 = 13^2$ holds true.

If you had instead climbed directly up the mast above the hook, you could go up the full 13 steps, because x and y would be zero. This is the maximum height you can climb while holding the rope and is represented by $0^2 + 0^2 + 13^2 = 13^2$.

Moving through all combinations of x , y , and z that satisfy this relationship, you trace out the surface of a sphere. What you'll notice is that the farther on the deck you get from the starting hook before climbing upward, the lower you will be able to climb.

What does the safety sphere look like in Foursville? Our new friend w joins the party. We now need to share our

length of rope between four friends instead of three. Just like sharing a delicious cake with party guests, if everybody wants some, then there will be less to go around. Everyone will get a smaller slice if it's spread evenly, but there's always the possibility that one particularly selfish guest could hog the entire thing to themselves. The relationship governing how the rope will be shared is now $x^2 + y^2 + z^2 + w^2 = 13^2$.

For every combination of x , y , z , and w that satisfies the relationship, you have a point on the **hypersurface** of a four-dimensional hypersphere. Forget the safety sphere—we now have a *safety glome*. If you extend the rope fully in the positive w direction, you have moved 13 steps *ana*, without moving in x , y , or z . But if you move in x , y , or z first, the rope has less reach to extend into the fourth dimension.

The three-dimensional safety sphere contained within it every possible orientation of a circle of radius 13. Likewise, the four-dimensional safety glome is made up of every possible orientation of a three-dimensional sphere of radius 13. The boundary of the safety glome is itself a three-dimensional surface, just as the boundary of the safety sphere was a two-dimensional surface. The space contained by the safety glome is a four-dimensional **hypervolume**.

Though our minds might struggle to picture this situation fully, mathematics gives us a way to grasp the unimaginable. You can extend this method to find the zone of safety traced out by a 5, 10, or even 1,000-dimensional hypersphere. In the latter case, you might need to get creative when coming up with names for all 1,000 of your variable friends. In the examples below, instead of naming each one with a new letter, I simply refer to each variable as x_n where n is the number of dimensions into which the sphere extends. So x_2 replaces y , x_3 replaces z , and so on.

Hypersphere relationships:

5-dimensional hypersphere:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = r^2$$

10-dimensional hypersphere:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \dots + x_{10}^2 = r^2$$

1,000-dimensional hypersphere:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \dots + x_{1000}^2 = r^2$$

Mathematics truly can transport us across dimensions in an instant.

We can also use mathematics to find the volume of a hyperball. It can be useful to sometimes call these higher-dimensional volumes *hypervolumes*, as they remind us we are dealing with something unfamiliar. The volume of a hyperball is the space enclosed within a hypersphere.

For an n -dimensional ball, here's the relationship between the volume V_n , and the radius r :

$$V_n = \frac{2\pi r^2}{n} V_{n-2}$$

It is a recursive relationship and builds upon the volume of the ball two dimensions lower than the ball in question (V_{n-2}).

Let's use this relationship to find the volume of a three-dimensional ball ($n = 3$), the kind we are already very familiar with. The symbol π (pi) is a friendly mathematical constant whose value is approximately 3.14. Now all we need to complete the equation is to find V_{n-2} , the volume of a one-dimensional ball, since $3 - 2 = 1$.



A world of one dimension is a single line, and a ball in that space will fill the line with a width of twice the radius, so the “volume” of that ball is simply $2r$.

The volume of a three-dimensional ball thus becomes:

$$V = \frac{4\pi r^3}{3}$$

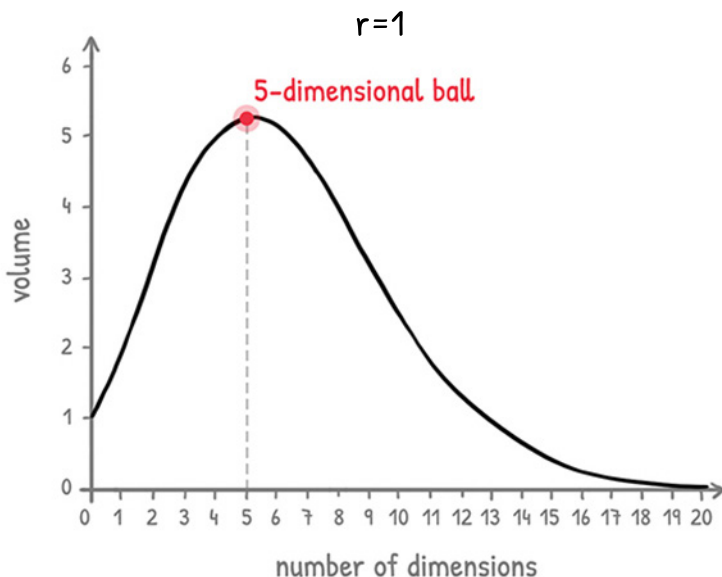
To get a numerical value from this relationship without worrying about the pesky little r variable, we can just set the radius of the ball equal to 1. Now we have:

$$V = \frac{4}{3}\pi \approx 4.19$$

Below is a table showing the volume of hyperballs in ten different dimensions, all with a radius of 1.

Dimension	Formula	Volume when $r = 1$
0	1	1
1	$2r$	2
2	πr^2	3.14
3	$\frac{4}{3}\pi r^3$	4.19
4	$\frac{1}{2}\pi^2 r^4$	4.93
5	$\frac{8}{15}\pi^2 r^5$	5.26
6	$\frac{1}{6}\pi^3 r^6$	5.16
7	$\frac{16}{105}\pi^3 r^7$	4.72
8	$\frac{1}{24}\pi^4 r^8$	4.06
9	$\frac{32}{945}\pi^4 r^9$	3.30
10	$\frac{1}{120}\pi^5 r^{10}$	2.55

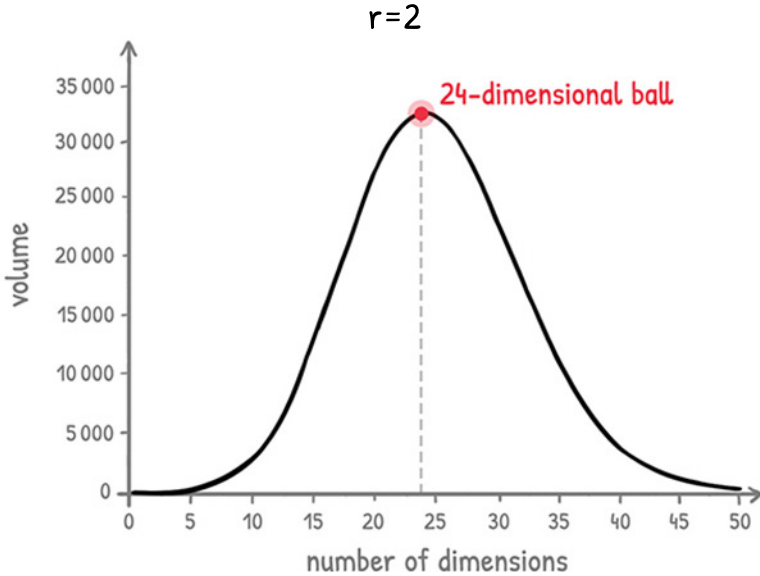
When plotting these volumes on a graph, a curious trend emerges. Look closely and you will see that the volume steadily increases as the dimension of the hyperball increases, then peaks in the fifth dimension before dropping off.



According to this graph, hyperballs with a radius of 1 have the most volume in the fifth dimension. That is where the juiciest hyperpeaches can be found! The caterpillars are probably having a raging tea party there.

It seems strange that the volume of a hyperball could decrease as the dimension increases beyond the fifth. In fact, the volume of an infinite-dimensional ball is zero, and infinite-dimensional hyperpeaches offer no sustenance at all.

However, the peak on the graph does shift if we change the radius of the hyperball. The volume of a hyperball with a radius of 2 peaks near the 24th dimension.



The relationship between the radius of a hyperball, r , and the dimension n_p in which its volume peaks is as follows:

$$n_p = 2\pi r^2 - 1$$

Rearranging this relationship, we can find the radius of a ball that would have its volume peak in any given dimension:

$$r = \sqrt{\frac{n_p + 1}{2\pi}}$$

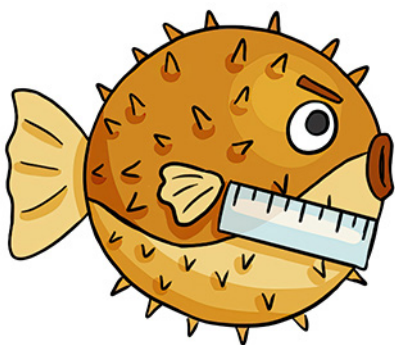
So, what would be the radius of a ball whose volume peaks in the third dimension?

Using $n_p = 3$, the resultant radius is $\sqrt{\frac{2}{\pi}}$ which is approximately equal to 0.7979. A ball with this radius will have less volume in any of the higher dimensions and in any dimensions lower than three.

This special constant is rather curious, and I wonder if it reveals any deeper truths about geometry in our homeland. I haven't provided any units for the radius

or volume—whichever you choose to use for one will apply to the other.

For now, though, it's just a useful fact that I use when I go fishing. There are restrictions on the size of fish that interdimensional anglers are allowed to take home. Hyperspherical pufferfish are the most common species to catch. When I reel one in, I take a measurement of its



radius. If I'm fishing in the third dimension, then the radius needs to be close to 0.7979 inches for me to be allowed to take it home. Such a fish is at its maximum volume.

If the radius of the fish is something else—say, closer to 1 inch—then I will leave

it to continue swimming between dimensions and allow its volume to reach its full potential by the time it reaches the fifth dimension. This may seem counterintuitive, since it's a larger fish that I'm releasing, but it's about the potential it has to be even larger in another dimension. Releasing juvenile hyperfish helps ensure the long-term sustainability of the species.

A GAME OF DODGEGONGOL

We've been cooped up on the boat for a while, so we should stretch our legs. Here in Foursville, familiar games take on new rules. Consider, for instance, our traditional jungle gym, now transformed in 4D. It's not just up, down, left, right, forward, or backward anymore; there's an entirely new direction to explore! Imagine the excitement of a race where I call out four coordinates instead of three.

Ready? X1, Y3, Z2, W5, go!

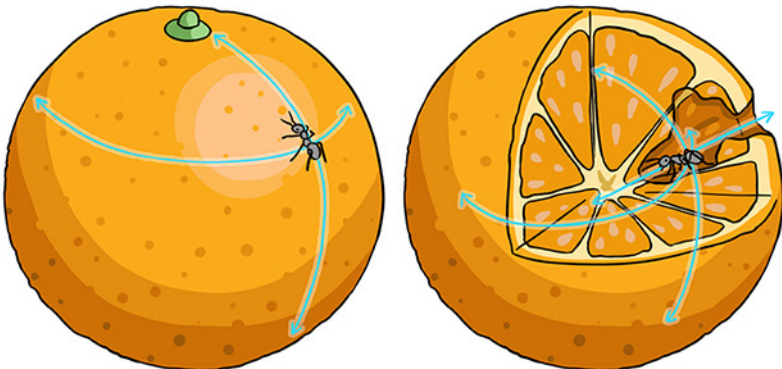
(You will find it pretty hard to play this version, since you will sometimes bump into someone that you couldn't have seen coming.)

Now, let me introduce you to one of the most popular sports in Foursville: dodgegongol. Think of it as dodgeball, but with a twist that only the fourth dimension could provide: Here, we play with hyperballs. The gongol is the 4D hyperball, and we can refer back to pages 134–5 if we want to find its volume.

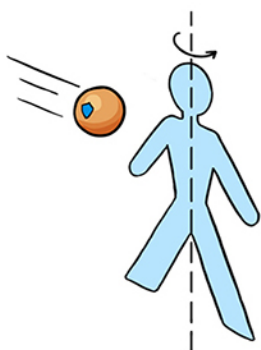
Mathematicians haven't gone to the effort of giving funny-sounding names to all the hyperspheres and hyperballs; instead, they are happy to refer to them as ***n*-spheres** and ***n*-balls**. Spheres have one less dimension than the ball they contain.

Take the case of a 3D spherical fruit, such as an orange. An ant walking on the skin of an orange is walking on a two-dimensional surface because the ant can move in two independent directions, called **latitude** and **longitude**. The skin of an orange thus approximates to a 2-sphere.

An ant that tunneled through the skin and exists inside the orange now lives in a higher-dimensional space of three dimensions. They can traverse latitude, longitude, and a third juicy direction that moves them closer to the center of the orange. The interior of an orange is a 3-ball.



A gongol is a 4-ball, and its hypersurface is a 3-sphere. An ant walking on the hypersurface of a gongol could move in three independent directions: latitude, longitude, and something else—call it **glomitude**. We can count our lucky stars that the pirates we encountered earlier didn't hurl any cannongongols at us! In Foursville, sports like footgongol and netgongol are the rage, but the crown jewel is dodgegongol.



Playing dodgegongol here, as three-dimensional beings, we have an unusual advantage. Our 3D bodies can “hide” by rotating, rendering us nearly invisible from certain 4D angles. Imagine, if you will, 2D beings in a 3D game of dodgeball, skillfully turning their paper-thin bodies to evade throws. We’re quite the sensation in the interdimensional dodgegongol league!

But it’s the 2D and **1D** teams, nearly impossible to spot, that truly excel. And the reigning champs? The **0D** team. They are mere points in space, so finding them is like trying to spot a bad pixel on a giant movie screen. Knocking one out, which often only happens by mere chance, is a feat worthy of the coveted Goofgongol Prize.

The offside rules take a bit of getting used to. They’re the cause of many arguments, and the referee often has to break up fights between teams quarreling over the measurement of the game’s boundaries. Sometimes, 2D creatures try to argue that anyone stepping outside of their plane is “cheating” and get upset when they are struck from “above” or “below.”

As we jump into a game, remember: In 4D dodgegongol, always check that extra direction. Ah, got hit by an unseen

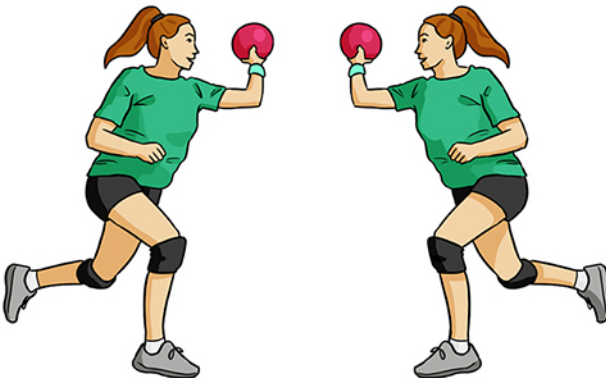
ball, did you? No worries! With a bit of practice in orienting yourself in 4D, you'll soon be dodging like a pro.

When we turn here, the world becomes a mirror image. Our own bodies remain as familiar as ever, but relative to our environment and other players, we can suddenly become flipped. You'll have to try and rotate yourself for an optimal throw with your dominant hand.

Here, catch this ball! Oh dear, you dropped it. Do you happen to be right-handed back home? I think I may have accidentally thrown the ball to your left hand. I couldn't tell which was which!

I recommend that you wear an armband on your right wrist so that your teammates can tell which one to throw to. Since you are right-handed, your right arm contains stronger muscles and has greater hand-eye coordination than your left, regardless of its orientation in space. We might want to pass to each other to better position ourselves for an attack, and it would be a shame to fumble it.

One more thing to be careful of when playing games in four dimensions is to not dislocate your joints! Think about a Flatling bending their arm in front of them to deflect a ball, their shoulder is not made to bend that way! Likewise, you should be gentle with your movements and rotate your body

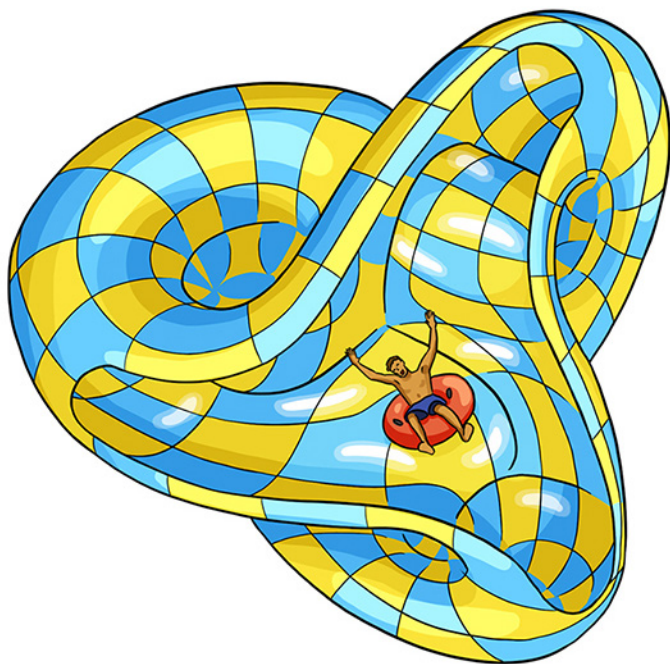


rather than overextending your joints in the new direction. Unless you are also hypermobile.

HINTONLAND

Now the time has come for the stop that many of my passengers onboard have been anticipating: We are at the gates of Hintonland. It's a large theme park with rides, games, and 4D creatures dressed up in costumes. The park is named after Charles Hinton, to honor his passion for seeing fantastical ideas come to life. You have a few hours to explore, so try to take it all in. The place may seem magical, but remember that it's because you can only ever see a small slice of the greater happenings.

Take a look at that waterslide. It's the most bizarre shape you've ever seen. You ride in an inflatable torus that glides upon a slippery two-dimensional surface. This is a ride

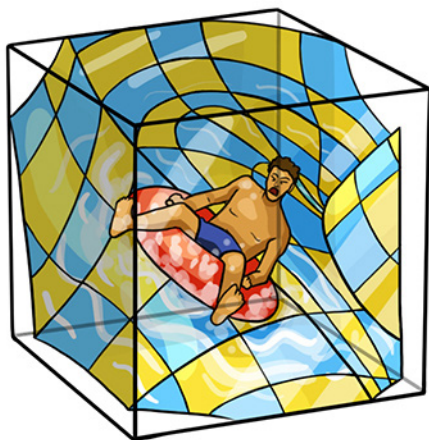


made for tourists from the third dimension. Changing the distribution of weight allows riders to have some control over their movements. Riders sometimes disappear from our view entirely, since the ride twists through all four dimensions. The staircase to the top is hidden until we rotate our view in the ana direction.

There are multiple funnels, bowls, and tunnels to navigate. And the weirdest part is these shapes are all connected, without any dead ends. In Threesville this wouldn't be possible. The neighboring waterslides are just as wild. In fact, any imaginable 2D surface can be built in Hintonland without intersecting itself.

Give it a try! Take it easy, though: The multi-axis movements could leave you feeling unsteady afterward. Remember to smile at the camera, which will take your photo on the way down.

There's a room at the end of the ride where you can purchase your photograph if you wish to take it home with you. The camera is a polaroid that prints the photos on the spot. Back home, a 3D camera prints photos onto 2D paper. Here, the 4D camera prints photos onto a block of light-



sensitive material. Each rider's photograph is a colorful cube encapsulating their happy—sometimes fearful—experience.

Let's now join the line for my favorite ride. This one is called "It's a Flat World," and riders sit in a little boat that travels slowly through a castle, while hundreds of small

animatronic creatures sing and dance. Their backdrop is a miniature village with houses and buildings representing every corner of the world.

The animatronics and their village are all constructed to be 3D. Since all 3D objects look flat to a 4D creature, the theme song “It’s a Flat World” makes a lot of sense. While we don’t have 4D vision, and, therefore, don’t see it in the same way, we must piece together a sense of what it could be like for a Foursville native to look upon a 3D world as we ride. By rotating our view, we can piece together what a 4D eye might see with one glance.

Even though the doors and windows of the miniature houses are closed, we can see inside. As the little boat weaves around, we can see inside the entire house. Little animatronic residents wave out from inside closed-off rooms. We see toy money, fake gems, and other treasures hidden away inside sealed vaults. We see the inner workings of the robots, including all the wires and components hidden behind their brightly painted exteriors.



A maintenance worker kneels beside one of the miniature buildings. She reaches in and removes one of the little sleeping animatronics from its bed, without opening any

of the doors and windows of the house. I see your stomach churning. Are you afraid this might happen to you back home?

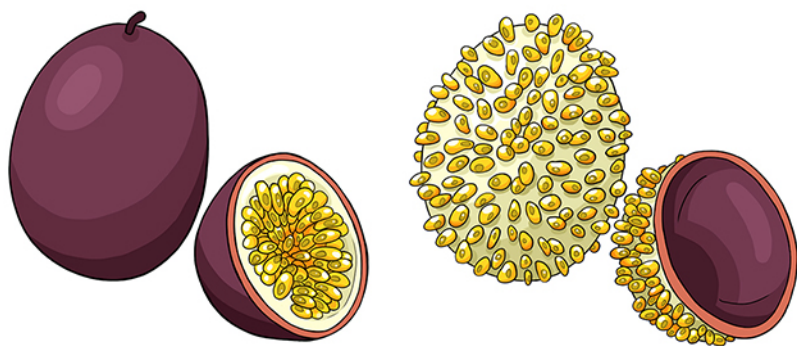
Let's settle those nerves with some pancakes—they are sure to make you feel better. The parlor outside the ride serves the most delicious pancakes in all of Foursville, and, in fact, I think they might be the best you've ever encountered. They are much fluffier than back home!

A pancake is made by covering the flat surface of a pan. Back home on Sunday mornings, I cook two-dimensional round pancakes. My pancakes do have a slight 3D thickness, though; otherwise, no one could pick them up to eat. Here, as you may have guessed by now, the pancakes have one extra dimension. A pan here also has a flat bottom, but the shape made by covering it in batter is a cylinder. It's a 3D shape but also extends slightly into the fourth dimension so that all the patrons here can eat them!

In the center of our table is a stack of passionfruit. They will make for a delicious accompaniment for our pancakes. These passionfruit are the three-dimensional kind, imported from sunny Threesville. This parlor does cater to tourists after all. You will see that eating 3D passionfruit while in a four-dimensional space comes with its advantages. One is that we don't need a knife to open the fruit and get the seeds out. We can simply use the extra dimension to turn the passionfruit inside out.

The passionfruit has a spherical shape, and its inedible exterior has a bit of flexibility. In three dimensions, you would not be able to turn a passionfruit inside out without creating an incision. Here, though, despite being a three-dimensional object, the passionfruit can move in all four dimensions. Turning it inside out is easy; it's similar to turning a rubber band inside out. You just twist the whole thing through the extra dimension.

The passionfruit's juicy interior is open to the fourth dimension, so unfortunately, the juice will have already escaped before we can get to it. The seeds are still in there, though, held to the skin. Someone viewing the passion fruit from its open sides could actually access the seeds without any fancy tricks at all, but turning it inside out spreads out the seeds and makes them easier for us to eat.



Now that our bellies are full, I propose we keep moving toward our next adventure. Let me introduce you to another friend of mine, a brilliant mind who has a unique talent for making the concept of a fourth dimension feel almost tangible.

THE ART HOUSE

One thing I like to do while on holiday is to go shopping. Traveling to a unique place means I get to gift some very unique souvenirs to my friends back home. For a long time, I have been collecting snow globes from the places I travel to. Foursville's craftspeople are very good at making snow globes because they are able to place pieces inside the glass bulb through the fourth dimension. When I take these home to Threesville, people always question, "How did they get that inside the glass?"

On our way to meet my next friend, let's take a walk down the Foursville Pier and look at some of the souvenir shops. Here you'll not just find the most impressive snow globes you've ever seen, but you'll also find magnets invisibly suspended on a hidden metal wall, shirts that say



"NOI2NEMIDHTRUOF EHT HƏUORHT DƎPƆI 2AW TRIH2 2IHT," and hats that read "ƎRTUAEƆD DƎ" and "2JJIV2RUOF." There's also an array of delicious, frosted doughnuts that have a few more holes than those back home.

Snow globes are only one of many things that I collect. Most of my recent efforts have been spent on collecting something more sophisticated: art. You may have noticed some of my prized pieces lining the walls aboard this ship. The pier is also home to the shop of my artist friend, Forthby Wonders.

Forthby's shop is located near the end of the pier, next to a ride called the 7D Experience. On its other side is a left-handed store selling regular 3D right-handed items that have been flipped here and are, thus, useful for left-handers back home. You could just rotate your own objects while here, but these items, including scissors, potato peelers, and spatulas, are pretty. Shops like these are how you can tell this wharf is made for tourists.

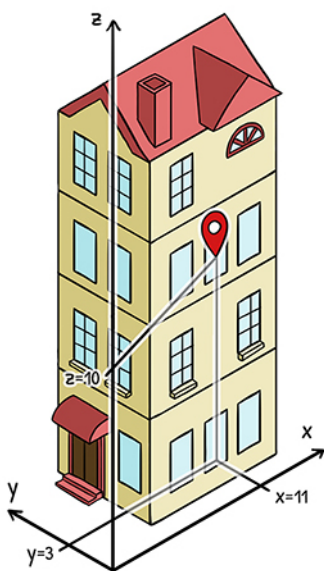
Overhead, a sign on an arch reads "Forthby Wonders' Art House." Follow me inside. Forthby's space is more of a museum. He collects and creates art because he is passionate about preserving the culture of different dimensions. He has convinced me that art is a useful

way to understand the lives of people from dimensions we cannot comprehend. Forthby is a collector first and foremost, but he occasionally sells pieces to those he deems worthy: individuals who show a true understanding and appreciation of the art's deeper meaning.

Now, I must warn you: Forthby is quite the recluse. He doesn't really like visitors, and when new people stop by he tends to move around in the shadows of the museum, just watching to make sure we don't damage anything. If you do see him, you'll love his hair. It's a wild mane, adorned with trinkets and artifacts from various dimensions.

The building is a beautiful space with high ceilings and artworks filling the walls. The floor on which we stand now is patterned like a kaleidoscope, in a mix of colors. The use of colors here is actually a clever way to organize the artwork. Let me explain.

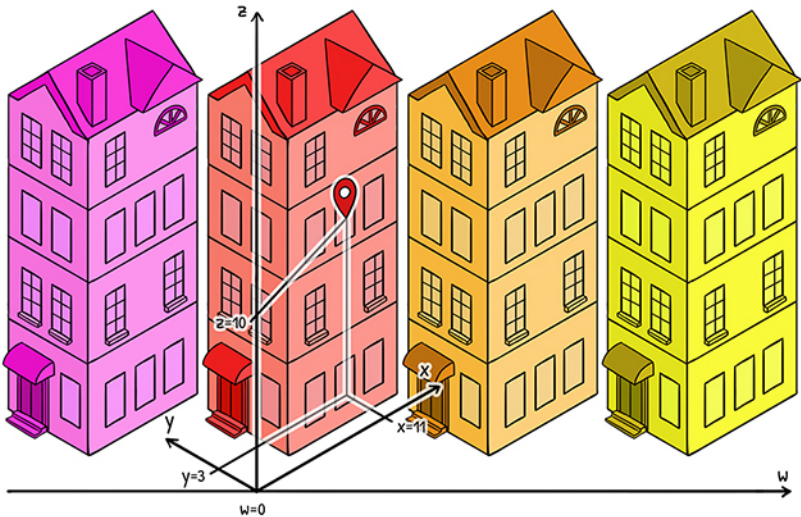
When navigating a world where an extra, fourth spatial dimension exists, we must find new ways to provide



directions. In Threesville, we give three pieces of information. For example, to tell a pizza delivery person where to deliver a pizza, we first get them to find the base of the building on a 2D map. This requires two pieces of information: how far north to go and how far east to go. Then we need to tell them how far up to go in the elevator—for example, to the 13th floor. These three pieces of information come together to pinpoint the exact delivery address.

However, when delivering a pizza in Foursville, they would need more information. Here, there are multiple versions of the 13th floor all occupying the same 3D space but separated along a fourth dimension. Each version of the 13th floor exists in a different ana-kata position along the w axis.

We can turn to colors as our guide to help distinguish the floors. In Foursville, we might instruct the delivery person to go to the “red 13th floor.” Here, “red” signifies a specific position along the w axis. In Forthby’s house, red is $w = 0$, orange is $w = 1$, yellow is $w = 2$, and so forth.



“I can demonstrate if you like.”

“Oh, that was Forthby’s voice! Hello, Forthby! Yes, please do give us a demonstration. My friends here would surely love a tour, too.”

“Hello, new friends, I am Forthby Wonders. Welcome to my art house. I have many rooms of art here, but you won’t need to walk far to visit them all. The rooms are stacked in four dimensions, and all you need to do is stand still in the central ballroom, where you are now. When I spin the color

dial on the wall, the lighting colors will change, and so will the artworks as the space shifts through the hidden dimension.

"Right now, you will see that the room lights are red. Here at $w = 0$ our first collection can be found in the center of the room. It is a cubism-inspired exhibit titled ' π -casso.' Cubism was an art movement that sought to represent the changing nature of space, time, and experience. Instead of depicting objects from one viewpoint, cubist artworks portrayed them from multiple angles simultaneously. This resulted in fragmented forms with overlapping lines and shapes.

"In this special exhibit, instead of paintings, you'll see sculptures that bring some cubist characters to life. Watch how, with the slightest twist of the color dial, these sculptures change. Cubist paintings often have overlapping lines on a flat canvas, but our sculptures show that, with four dimensions, these lines and shapes don't overlap at all. The subjects are beautiful creatures of four dimensions who display multiple emotions at once.



"I will leave you to explore the rest of the floors with your captain at your own pace. At $w = 1$, you will find the orange room, where you'll meet Wrosie, our artist-in-residence. She's known as a '**Worldline** Weaver' and will be performing her next live show in 20 minutes. Farewell!"

WORLDLINE WEAVING

We thank Forthby for his introduction to the Art House. We have some time to burn before Wrosie's show, so let's talk about its theme. So far, we have spoken about the fourth dimension being one of space. In Foursville, time still exists. If you'd like to call time a dimension of its own, here it could be called the fifth dimension. But names and labels don't matter too much. We could equally call time the first dimension, and this strange new realm of perpendicular space could be the fifth dimension. It doesn't matter because the effect would be the same: There would still be a total of four spatial dimensions that march forward in time.

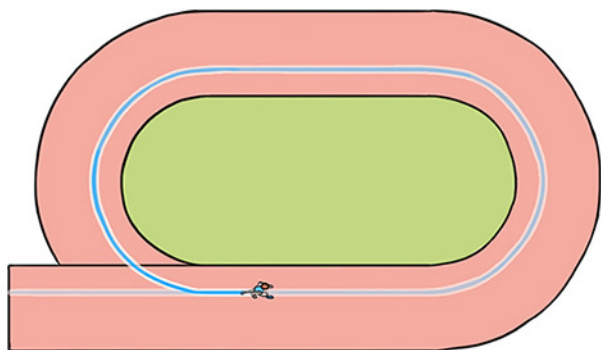
The interplay between time and a fourth spatial dimension is quite interesting because they share some similarities. A sphere that starts as a dot, then steadily grows to maximum size before shrinking away, is a phenomenon we've met in this book and blamed on a 4D hypersphere passing through our 3D world. The same sequence could, however, also manifest purely as a result of the way a 3D shape changes in time. A tomato hanging from a vine starts as a tiny dot but grows and swells to a maximum size over time. As time progresses, it will lose mass and wilt away to nothing.



Let's consider snapshots of our world at different instances of time, each only infinitesimally separate from the next. They stack together not to form a new spatial dimension but to construct a time dimension. Tracing the path of an object through time draws out something called a *worldline*.

Worldline diagrams are powerful tools, revealing the progression of events over time. A worldline is like a long-exposure photograph of someone holding a light in the darkness—it's motion forming an illuminated trail. If time didn't erase our past, we would leave such trails.

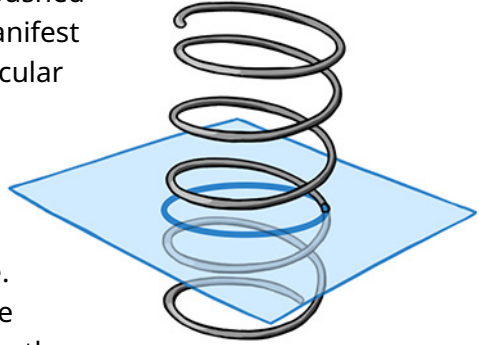
Picture your current trail. Perhaps it's a still image of you, lying comfortably in your bed, engrossed in this book. Now, envision a runner's trail circling a track, forming a loop. Or a car navigating city streets, its path weaving through the urban maze.



Interestingly, every atom in our bodies gets replaced approximately every seven years. So, instead of a single,

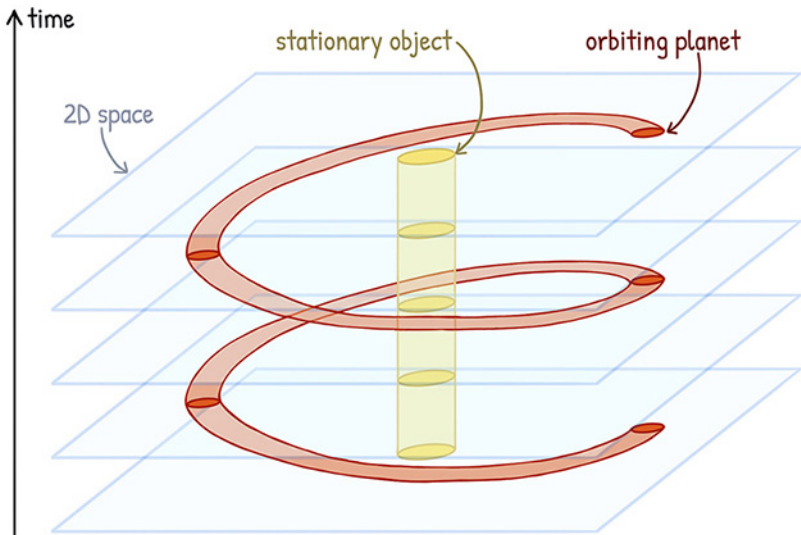
continuous trail, we might leave behind an intricate braid of worldlines, each representing an atom's path.

A 3D spring that was pushed into a 2D plane would manifest as a dot that moves in circular motion, either clockwise or counterclockwise, depending on if the spring was pushed from above or below the plane.



A dot moving in a circle could also be attributed to the orbit of a flat planet moving over time under the influence of gravity.

Leaving a trail through time, the orbiting planet would form a worldline spiral shaped just like the 3D spring. Any movement in 2D creates 3D shapes out of worldlines. Likewise, any movement in our 3D world forms 4D worldline shapes.



Visualizing a fourth spatial dimension might work best if you connect it with a time experience in three dimensions. The best way to visualize a tesseract, for example, may be to imagine a cube moving over time and tracing out that higher shape. If you've seen a sphere grow and then shrink (maybe while playing with a balloon), then you've seen a hypersphere, just spread over time instead of over a fourth spatial dimension.

The idea of four-dimensional creatures scares us because, if they wanted to mess with three-dimensional creatures, they could. They could travel *ana-kata* and appear in a closed room or steal gems from our locked safes. However, these feats could also be achieved by a regular 3D creature that can travel through time.

Such a time traveler could appear in a closed room by traveling to the future, when the room no longer exists, entering that space, and then traveling back to the present. They could steal gems from a safe by traveling back in time to a moment when the box was unlocked, moving them somewhere else, and then traveling back to the present, acting none the wiser.

If time is a dimension that persists, you could imagine observing all of time and space laid out before you. Yet, our experience is constrained to the fleeting moment of now. We are trapped on a ceaseless forward current in the river of time. This notion raises a more profound question: What if our passage through time isn't what it seems? What if, instead of us moving through time, we are static, and time itself is moving through us?

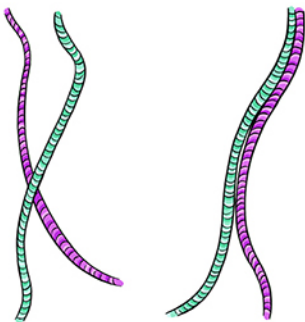
The concept of a worldline, representing an object's path through spacetime, being a fixed and solid structure, introduces a mind-bending perspective. If this structure is pre-made and unchangeable, moving through our dimension

like a spring through a 2D plane, then every moment of our lives is predetermined. It suggests that our lives are like beads threaded on a string, part of a worldline blanket that's already been woven.



This challenges our beliefs about free will and the power of our own decisions. It seems to describe a universe where the future is just as set in stone as the past, and our perception of choice is merely an illusion. The implications of such a worldview are vast and, for many, unsettling.

Let's head to the orange room at $w = 1$ and catch the live show. I think you will appreciate it—Wrosie the Worldline Weaver is amazing. Watch the colors of the room change to orange as we move there. I can see Wrosie sitting at a desk by the back wall. I don't want to disturb her, so I'll let you read the exhibit sign next to her.



Worldline Weaving is an ancient art practiced by a few skilled individuals. Here, in a world of four dimensions, a weaver can manipulate the worldlines of 3D creatures, essentially controlling how their lives will play out. By bringing two worldline threads

together, a weaver can ensure that two individuals will be certain to meet, and by pinning the threads together, the individuals can be certain to stay together in physical space. This art installment demonstrates the interwoven nature of 3D life.

The first time I saw worldline weaving, I was plunged into sadness. I felt like my life was meaningless because I saw that, to 4D creatures, our lives are nothing but a tabletop art project. But I've worked through those feelings and now feel at peace with the thought. I like to believe that we still have control over our choices and destiny and that, while the universe might have its patterns, we are not mere puppets performing a predetermined dance.

This sense of autonomy, of being able to steer our own course, has been central to our human narrative throughout history. Our stories, art, and daily decisions celebrate the triumphs, regrets, and randomness of choice. Regardless of where the truth lies, between strict determinism or the spontaneity of free will, our belief in our own agency remains a cornerstone of our human experience. As you can see in some of the art pieces on display here, and in other art you have seen in your own life, the balance between predestined paths and chosen detours makes our existence beautiful.

As the worldline weaving demonstration winds to an end, we move to the next room of the Art House. The lights change to yellow as we enter $w = 2$.

Guest Chapter by CodeParade**THE GAMES ROOM**

Do you see that strange arrangement of floating limbs hovering before us? That's another friend of mine who will show us around this room.

"Hello and welcome! I am ... Oh dear, please excuse me!"

It looks like they need a moment to orient themselves. Watch as the limbs shift and grow into the body of a man.

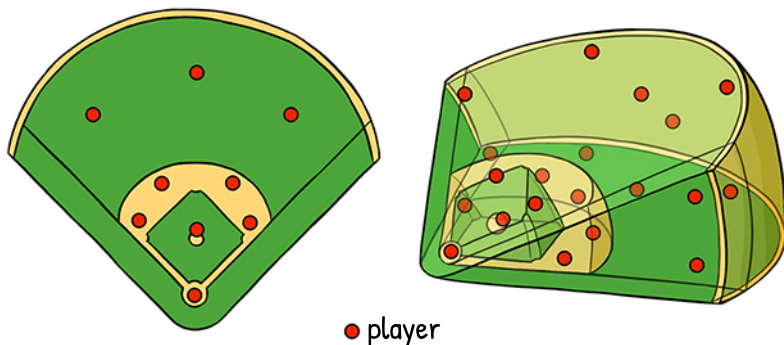
"I do apologize. We Fourthlings must always remember to align ourselves with our lower-dimensional guests. It's good manners!"

By now, you might have grown accustomed to such sights, but it still feels shocking, doesn't it?

"As I was saying, I'm Coach Clifford Cayley. I'm a curator for the sports exhibit here at the Art House. Foursville is known all around for its world-class sports teams and athletes. Come, follow me!"

He's leading us over to a glass case leaning against the wall in the corner of the room. Inside, we can see what appear to be autographed baseballs on little stands and a large bat on a miniature podium.

"Ah, yes! Basegongol. A Foursville classic. It's not too different from the 'baseball' you already know. But our fields are much more spacious than yours. In your three-dimensional baseball, nine players need to spread out on



the field for the best chance of catching the ball. They only need to worry about running to the front and back and side to side of the field. But in basegongol, we don't just have the infield and outfield, but also the anafield and katafield, so our teams are larger, with 20 players for an exciting game!"

The way Coach's fingers disappear and reappear means he is gesturing for us to follow him to the next display. "Now, let's not linger too long here. Our next few exhibits are interactive, so you'll have a chance to try out some four-dimensional sports yourself. This one seems very popular with tourists: It's a miniature golf course!"

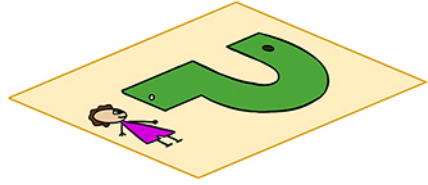
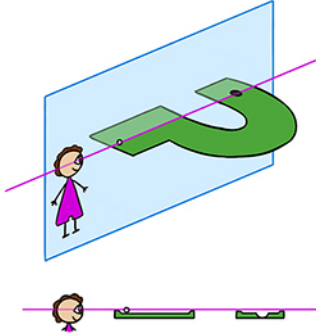
As we follow Coach toward the putting green, you'll see several twisting paths that seem to wind around the floor, each with different obstacles and designs.

"Go on and grab one of those multicolored balls and putters lined up next to the course. We are going to try out the game!"

You've lined up to take a shot, but I can see that you're just as confused as I was the first time I visited this course. It looks like our starting position just leads to a dead end! There doesn't seem to be any possible path to the hole.

"Indeed, this course would be impossible if you just relied on your three-dimensional perspective. But for anyone in Foursville, it would be completely natural. As your coach, I will help you see it more as we do; it just requires a quick shift of perspective.

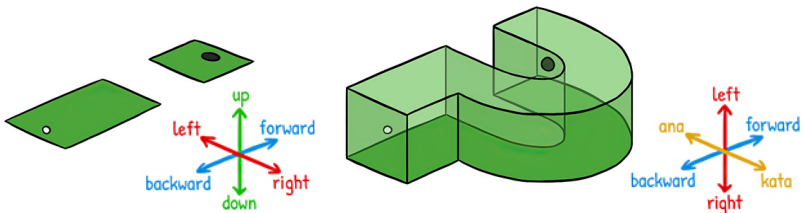
"You see, miniature golf in your world is almost entirely a two-dimensional experience because the ball will rarely leave the ground. So, if you wanted to show the sport to a Flatling, you might as well ignore the height dimension altogether, so they could see the entire course in a more natural way. This would require them to lie flat on the ground themselves, crawling around to plan their shots, but it's certainly a compromise worth making!



"So now let's do the same trick for you. I'll help you tilt your heads back into the kata direction so you'll be able to see everything on the floor completely clearly, with only the minor inconvenience of losing all information about the heights of things. Again, it's a trade-off you must make. I apologize if this tickles a little."

You'll feel the dizzying sensation of vertigo as the room starts shifting and warping in strange ways. The ceiling will disappear, and a rainbow of wall colors from the previous rooms will replace it. But don't worry about that, because once all of the twisting finally settles down, the shape of the course will suddenly click for you.

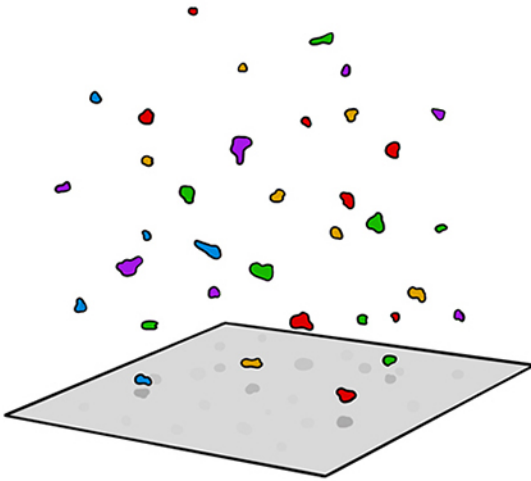
Do you see it now? There was only a simple bend to get around the wall in the ana-kata direction. So after only a few strokes, we're all able to sink the ball in no time at all.



After finishing up and reorienting back to the upright position, let's follow Coach to the next exhibit. You'll see a

large space filled with floating rocks that extends upward until it almost touches the ceiling.

"And now we come to rock climbing! It may look like the rocks you see are dangerously floating above your heads, but I can assure you they're safely attached to the wall just ana of you, so there are no risks to worry about—well, other than falling! But for that, I'll simply attach this harness to you and you're ready to go.



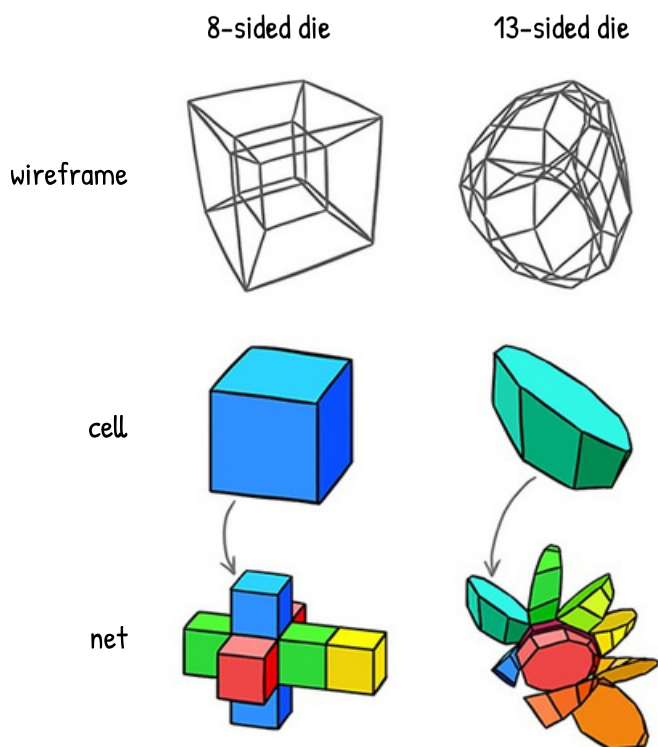
"The holds may look perilous, but don't let that deter you. Reach out for the nearest one and start pulling yourself up, one step at a time. Even if you've climbed before, this wall adds an entirely new dimension to the sport. You need to try and find a route in three dimensions, which is bound to result in some awkward positioning.

"It is physically exhausting, but you'll be glad to hear that our last stop will be a much more relaxing experience.

"After a workout of the body, it's a great time to work out the mind! Not all sports here are physical; games can also be just as competitive, if not more. Here's a typical setup

for a tabletop role-playing game. You see, these games rely heavily on dice rolls, so it's very convenient that we have so many options for them in Foursville. In fact, we have infinitely many! Need a seven-sided die? An 11-sided one? 137-sided? It doesn't matter because we can always make an entirely fair die with identical sides for any number you wish.

"How is that possible? With **gyrochora**! These are special shapes that exist in four-dimensional space, where every side is exactly the same, ensuring each face has an equal chance of landing face up. Whereas in your world the number of fair dice is limited to specific shapes like cubes or tetrahedrons, in Foursville—and in any even-numbered dimension—we can create fair dice with any number of faces.



"It's hard to imagine, but it might help to think about the situation in 2D. There, regular polygons (shapes with equal sides and angles) of any number of sides will make fair two-dimensional dice. If you can picture a Flatling rolling a shape with 13 equal sides, then it might not be too much of a stretch to see how extending the idea to four dimensions could work, with the sides twisting around to cover the whole shape.

"Well, friends, that wraps up my tour. I hope you've enjoyed this glimpse into the games of Foursville. Whether it's watching a game of basegongol, climbing a 4D wall, putting in mini-golf or even rolling a gyrochoron, there's always something exciting happening here. Best of luck on the rest of your tour—I'll be cheering you on from the sidelines!"

And with that, we say our final goodbyes and give a wave as we turn the color dial and quickly slide back into the central ballroom. From there, we depart the Art House and head back to our ship, ready to set sail for even higher spaces.

MIRROR LIFE

As we leave Foursville, it is essential to spray ourselves down for decontamination, much like astronauts and spacecraft are cleaned upon returning from space to avoid inadvertently bringing any alien life home. Human skin hosts countless bacteria, some of which accompanied us to Foursville. During our time flipping through the fourth dimension, some of this bacteria may have been transformed into its mirror-image

form. Taking these mirrored bacteria home to Threesville could change life as we know it.

Mirrored bacteria might sound harmless but are, in fact, a major existential threat to the 3D world. I'll explain why, but I must admit: Sharing this knowledge could be risky, especially if you're an evil genius in disguise.

As we discussed earlier in our voyage, many essential molecules for life on Earth exist in two mirror-image forms, referred to as "left-handed" and "right-handed" configurations. These **chiral** molecules are not interchangeable: Our proteins are made of left-handed amino acids, and our DNA is built using right-handed sugars.

Mirrored bacteria could dominate the ecosystem without natural predators. They'd only need sunlight to photosynthesize, but as they spread, they'd produce flipped sugars that regular organisms, including us, couldn't use for energy. These mirrored sugars and other resources would no longer provide sustenance to other creatures higher in the food chain, effectively starving ecosystems from the bottom up.

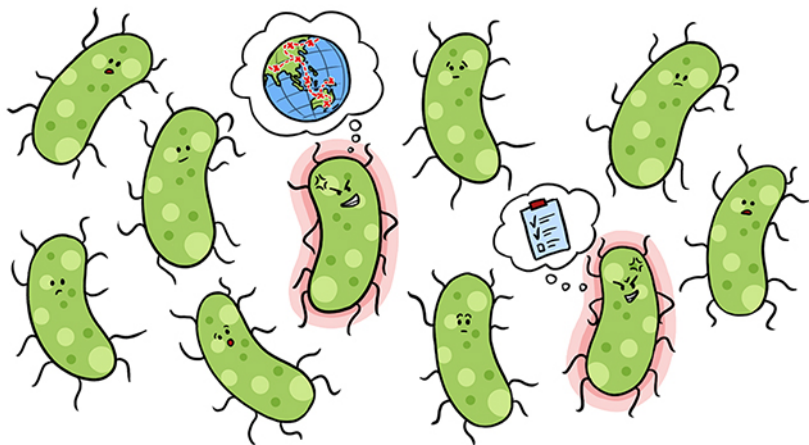
The human immune system (and those of other animals) would be defenseless against mirrored bacteria. Our immune systems rely on precise interactions between chiral molecules to identify and neutralize threats. However, mirror bacteria's flipped molecular structures wouldn't properly bind to our immune receptors or enzymes, rendering them effectively invisible to our defenses.

While evading our immune system, a mirrored version of a common bacterium like *E. coli* could survive and grow inside of us on non-chiral nutrients such as glycerol, which are the same when mirrored. Our lack of immune response could allow the mirror bacterial infection to grow extensively within the body.

Most existing antibiotics and vaccines are designed for chiral targets, making them ineffective against mirror bacteria. Developing new “antimirror” compounds and flipped vaccines would be essential—or we might need to import flipped medical supplies from Foursville.

Alternatively, we could employ a classic explorer’s tactic: introducing a new predator. Releasing a mirror phage—a virus that preys on bacteria—might keep mirrored bacteria in check. These mirror phages could replicate and destroy mirror bacteria, controlling their population. However, such efforts wouldn’t prevent the initial invasion of mirror bacteria from just spreading into a new environment.

So, as we shower off on the deck, we do our part to avoid triggering such a disaster. Yet, this doesn’t mean there’s nothing to worry about. Scientists back in Threesville are already working on creating mirror-flipped life in labs. They’ve synthesized flipped enzymes like polymerase, capable of producing mirrored DNA. Their goals are noble: to develop longer-lasting medicines that won’t be broken down so fast by the immune system. But stay cautious, my friend. Scientific voyages come with both wonders and risks, and it’s up to us to tread responsibly.



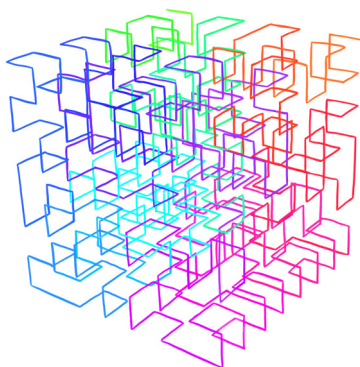
TWO PAPER CUT-OUTS

I lost you, though I'm standing still,
I turned my neck, things went downhill,
I didn't want this kind of thrill,
When I booked us tickets to Foursville.

I see other people's faces,
And unfamiliar spaces,
I see new shapes in new places,
They should have given us neck braces.

Like a lost Flatling, I am turning,
In 4D space, stomach churning,
I'm a little paper cut-out yearning,
To find you and know that I'm returning.

Scanning, I catch a glimpse of you,
Doubling back, you're now in view,
You grab my arm, I grip yours too,
And keep my eyes on yours like glue.

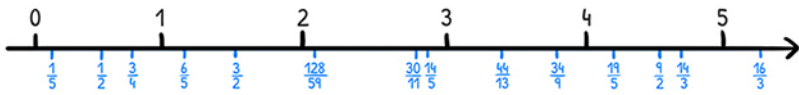


PART 4.5:

FRACTAL ISLAND

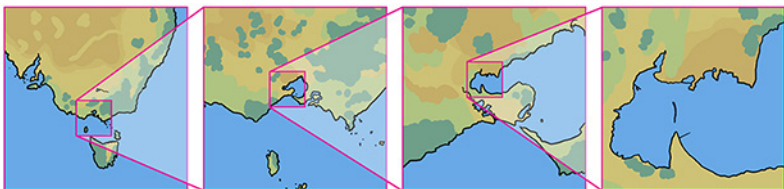
Throughout our voyage so far, **dimensions** have always been whole numbers. We've visited the worlds of **4D**, **3D**, **2D** and speculated about **1D** and **0D** places. But numbers are not always whole, and neither are dimensions.

Fractional numbers represent parts of a whole. They can be written as fractions, like $\frac{1}{2}$, or as decimals, like 0.5. These are numbers that exist between 0, 1, 2, 3, 4, etc., and give rise to the intriguing concept of fractional dimensions.



Fractional dimensions don't fit neatly into our standard understanding of space. They exist somewhere between our whole dimensions. This "in-betweenness" is what makes them a powerful tool for understanding the complexities of certain structures, particularly **fractals**. Fractals are detailed patterns that reveal more intricacy the closer you examine them.

As we sail the inter-dimensional seas, an island appears in front of us. Take a look at its coastline. At first glance, it might appear somewhat straight. As we get closer to our approach, you'll notice more nooks and crannies. We are now close enough to make out the shape of a person walking on the beach. From this vantage point, even finer details emerge in the rocky cliffs where water meets land.



This ever-increasing complexity at each level of observation is the hallmark of fractals: They consistently

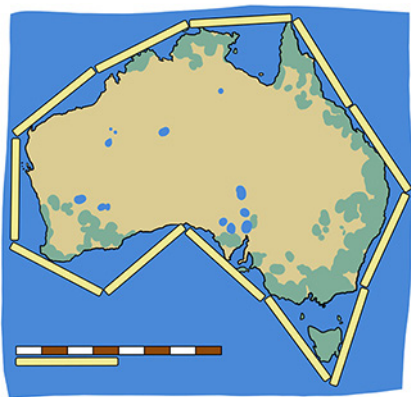
exhibit detail, no matter the scale. The idea of fractional dimensions helps us describe these patterns, since they don't fit easily into the way we usually think about shapes.

That person walking on the beach is my dear friend Mr. Mandelbrot; he is the coastguard of this island. His job is to measure the length of the coastline—something he thought would be an easy task when he first accepted the job. That was years ago, and Mr. Mandelbrot still walks the coast with a bag full of rulers.

The question "How long is a coastline?" has a surprisingly complex answer. The "length" can vary depending on the scale at which you measure it. If you measure using a

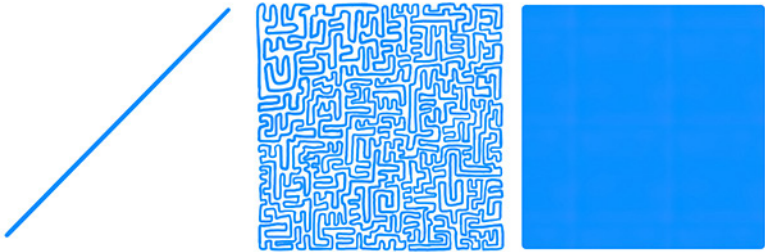
10-kilometer-long ruler, you'll get a much shorter length than if you were to measure it using a 1-meter-long ruler, as the smaller ruler can capture more intricate details of the coastline. But this raises a quandary: Which measurement of the coastline is the "true" length?

Mr. Mandelbrot has so far found that the coastline doesn't fit neatly into our conventional understanding of dimensions. Rather than being strictly one-dimensional (like a straight line) or two-



dimensional (like a flat surface), the coastline's dimensionality lies somewhere in between. This change in measured length at different scales means that the coastline and similar fractal structures have a fractional dimension.

Think about how the coastline of this island looks on a paper map. If the coastline were smooth, it would just be a line, and its dimension would be 1. If, instead, we considered the coastline to fill in an entire land area, then it would be two-dimensional. Coastlines are more complicated than a straight line, but they are not quite a full area, so their dimension is a fractional value somewhere between 1 and 2.



If you scrunched up the paper map into a ball, with folds at every scale, then you'd have made something with a fractional dimension. The surface area of the scrunched-up ball is hard to measure. The shape is more than a 2-dimensional surface but doesn't quite fill a three-dimensional volume. Its fractional dimension would land somewhere between 2 and 3.



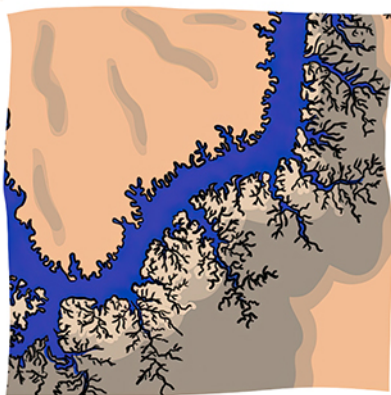
Coastlines aren't the only fractal-like wonder out in nature. Here are some of my other favorites:

Romanesco broccoli: Its spiral cones comprise smaller spiral cones, which in turn comprise even tinier spiral cones.

Ferns: Each small leaflet is similar to the shape of the larger frond.

Mountains: Mountain ranges viewed from a distance display repeating patterns, with smaller peaks resembling the shape of the larger formation.

Rivers and lakes: The main body of water branches into smaller streams, which divide into smaller creeks, and so on.

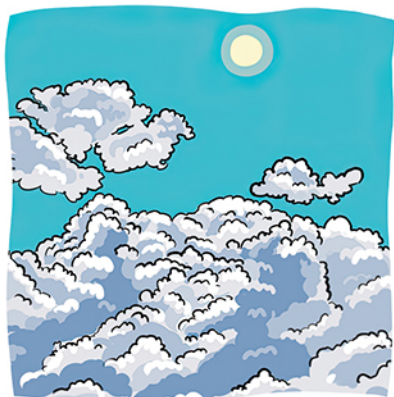
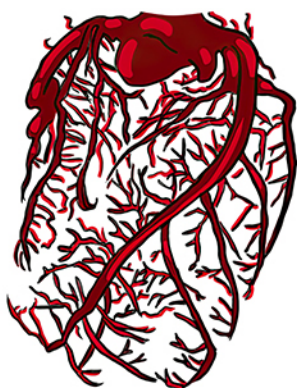


Blood Vessels: The circulatory system is a network of vessels branching out into smaller vessels.

Lightning: The jagged path of a lightning bolt often splits into smaller branches as it seeks the path of least resistance from a cloud to the ground.

Clouds: The fluffy edges of some clouds resemble intricate and irregular coastlines.

Coral Reefs: The growth pattern of some corals involves branches sprouting smaller branches in a self-similar arrangement.

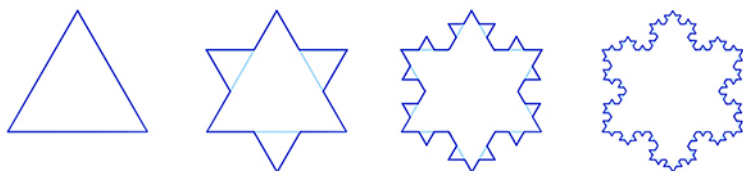


These “self-similar” fractals show that there’s a patterned regularity in the world around us, even in places where it might initially seem chaotic.

In the case of river systems, the repeating features at different scales can be attributed to the same forces of nature that also act the same at different scales. Take, for example, the process of erosion where water meets land. Erosion will work its magic on small or large features, leading them to resemble each other.

SENSATIONAL SANDCASTLE

Another example of a self-similar fractal is the Koch snowflake. To draw a Koch snowflake, begin with a triangle with three equal sides. This forms our base.



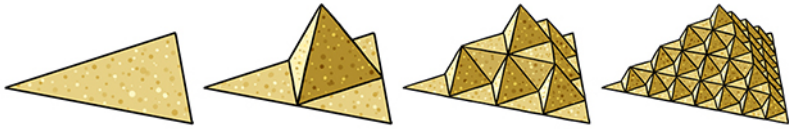
Then divide each straight line of the triangle into three equal parts. Replace the middle segment with two segments of the same length, forming a triangle without its base. This is the first iteration. Each line of the original triangle is now jagged.

If you repeat the last step, every segment will continue to be divided in the same manner. After each iteration, every line segment is replaced by four segments, each one third the size of the original.

The Koch snowflake, after infinite iterations, lies somewhere between a line and a completely filled shape, so its fractional dimension lies between 1 and 2.

Let’s do what we love to do more than anything: Let’s take this up a level. We are now so close to Mr. Mandelbrot’s

island that we can feel the sand grazing the bottom of our ship. Hop down onto the beach for a minute and try to make a version of the Koch snowflake out of sand. Ours should have a fractional dimension between 2 and 3. Let's call it a Koch sandcastle.



To get going, trace out a triangle, then build a large triangular pyramid that touches the midpoints of the three lines of the triangle. Here comes the tricky part: Carefully add a new, smaller, triangular pyramid on each triangular face that you can see. Continue this process until the sandcastle teeters on the brink of collapse.

After infinitely many iterations, we have a structure that isn't merely a 2D surface, nor is it a conventional 3D solid, but something in between. It has an infinite surface area but a finite volume. It's a detailed mesh with a fractional dimension between 2 and 3.

It might be too much of me to expect you to build this out of sand, especially on your first go! It requires a great deal of precision and patience. Beside you now is a man who possesses a great deal of both qualities: Say hello to Mr. Mandelbrot! He is working on his own sandcastle right now, one with a touch of four-dimensionality.

Mr. Mandelbrot's sandcastle began as one of the Koch sandcastles we just attempted. Then, Mr. Mandelbrot took this already intricate structure and extended it into a **fourth dimension**. He repeated the same iterative, fractal-building process we did for its 3D version, but in the fourth dimension. For every 4D face, he added smaller 4D protrusions, each iteration making the object more detailed.

His sandcastle is a fractal that has been extruded through a fourth spatial dimension, with every slice or **cross-section** of it (when viewed with 3D eyes) resembling the evolving iterations of our 3D Koch sandcastle. It doesn't fully occupy a 4D **hypervolume** in the way a **tesseract** does, but it is more detailed than a 3D volume. It has a fractional dimension somewhere between 3 and 4.

Up close, you can see Mr. Mandelbrot's nametag pinned to his shirt. It reads Benoit B. Mandelbrot. Fun fact, the *B* in his name stands for Benoit B. Mandelbrot. One last self-similar pattern to chuckle at. Let's wave goodbye to our new friend as we head away from the coastline.

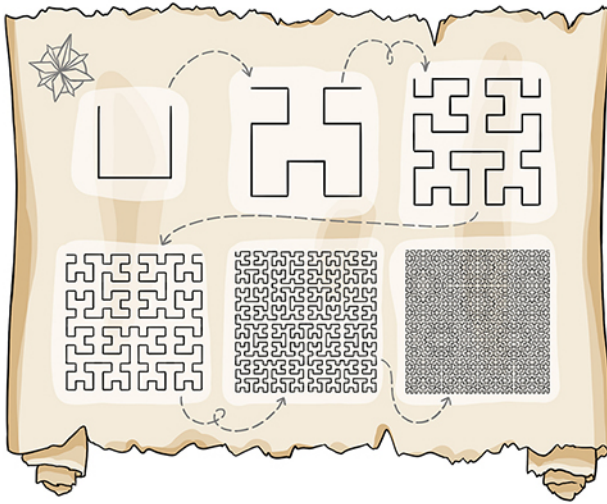
SPACE-FILLING FRACTALS

Have you met our navigator yet? She's tucked away in her cubby, charting our path and making sure we're pointed in the right direction. Come and meet Lacey the silkworm, our ship's navigational marvel.

Lacey protrudes silk in a line as she moves, leaving a one-dimensional glimmering trail. Look at what she's making now: it's part of a map.

She's made a U-shaped trail of silk. After that, she turns the corner and makes another U-shape, continuing around in a square, visiting each quadrant. Oh, but don't think she's done here—that shape is only the lower quadrant of an even larger shape with the exact same structure.

As we watch, Lacey meticulously covers an entire square area. She walks in short bursts, turning corners often but never doubling back. She has given the square what looks like a solid fill, but put your eye right up close, and you will see it is just a series of tiny U-shaped turns. She is creating a special kind of fractal called a **space-filling fractal** (which is sometimes called a space-filling curve).



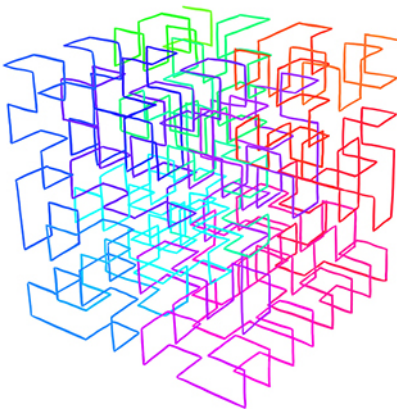
Done meticulously enough by a tiny silkworm, this pattern will cover the entire square, visiting every single point within it. It is a map of a two-dimensional space, made from a one-dimensional line. A ship merely needs to follow the line, and it will chart the entirety of **Twosville**. As I'm sure you know, we are interested in charting more than Twosville, and the same shape can be woven to fill a three-dimensional block.

What's fascinating about this map is that three-dimensional information has been turned into one-

dimensional information.

A point in 3D space usually requires three pieces of information to pinpoint, but now we only need one. That is, how far along the silk line have we traveled?

Think about our jungle gym experience. I previously gave instructions like X1, Y3, Z2 and watched

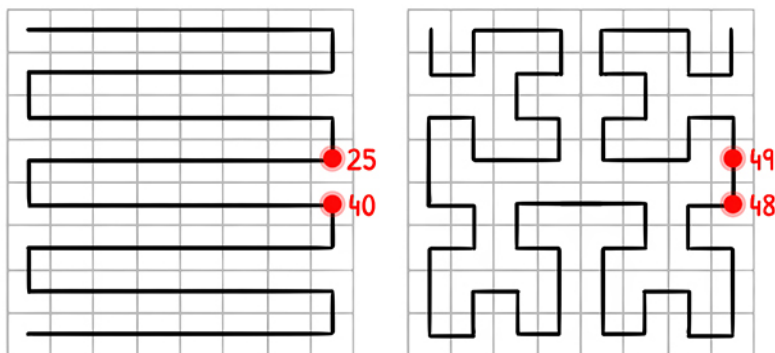


you all race to a single vertex. Instead, I could have given you each a copy of Lacey's map and then called out just one number. And the best thing is, this works in any number of dimensions!

In **Foursville**, instead of reading out four coordinates, I could still have specified a location with just one number. The map that we followed when charting the lands of Foursville was one of Lacey's creations. Her silk line folded so much that it became a four-dimensional space. Space-filling fractals are maps between a line and any n -dimensional space.

There are many species of space-filling fractals, and some simpler than others. The most basic one is formed by just scanning along each row of space. The problem with this method is that two points that are close together in space might end up having wildly different values on the resulting silk line.

That might be annoying because, for example, two spots on the jungle gym right next to each other might have values of 25 and 40, and that could throw you off your game. At least when using Lacey's map, points close together will always have somewhat similar 1D values (as shown on the right in the figure below).



Space-filling fractals are useful for more than charting the inter-dimensional seas. Back home in the “real world,” the act of mapping multi-dimensional information onto a one-dimensional line is a kind of pre-processing used by computers to make some information easier to work with. Even our DNA resembles a space-filling fractal in the way an intricate 3D molecule carries the information of a 1D genome sequence.

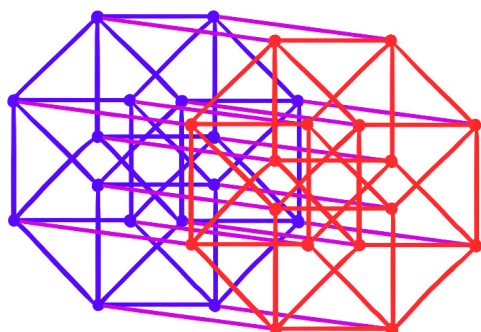
IN-BETWEEN

How I got here is a winding tale,
One with endless turns and detail.
I am between a pass and a fail,
Between snow and a coastal trail.

See, I'm not quite here,
And I'm not quite there.
Did I want to be a line or a square?
I told them all I didn't care.

In my quest to belong,
I stretched myself oh so long,
Trying to mimic a square's song,
But I see now that was wrong.

Before I went on the run,
I was tangled in the webs I'd spun.
I tried to be close to everyone;
Sometimes that leaves you close to none.



PART 5:

HIGHER

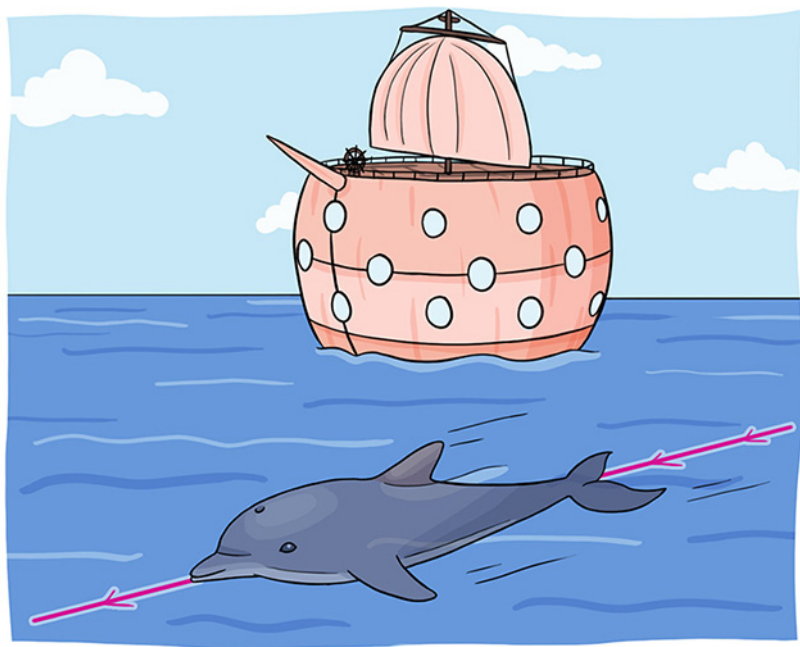
DIMENSIONS

The wind picks up, filling out our sails. We speed through the waves, sending splashes of water onto the deck. Peer over the side of the railing and you will see a marvelous sight. A dolphin gliding beside us, matching our speed like a guide escorting us back home. It looks like it's time for us to make another friend.

THE COMPACTIFICATION TRICK

"Hello, Kaluza!" I call down to her. She chirps back in response, looking up at us, her mouth wide with a happy smile. She looks at you and dives playfully underwater. I know what that signal means: She wants to show you some of her tricks.

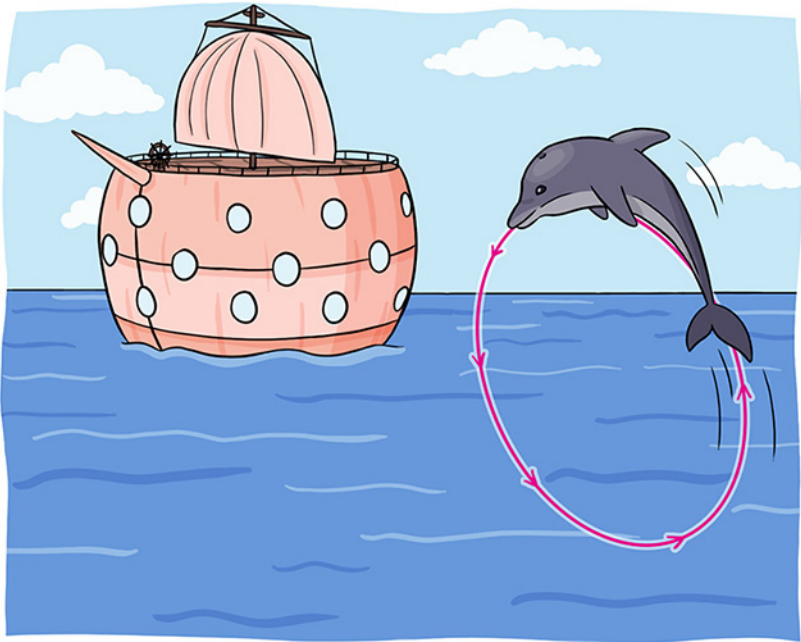
The first act in her routine is to show off her incredible speed. Water sprays out of the top of her head, and I know it's "game on." I adjust the sails so that we sail as fast as possible, gliding across the surface of the water. Watch



Kaluza, keeping her fin out of the water as she glides just under the surface. She travels in a straight line, channeling all her effort into her forward momentum. We are no match for her; she glides right past us and then jumps into the air in a celebratory move, happy to have beaten us.

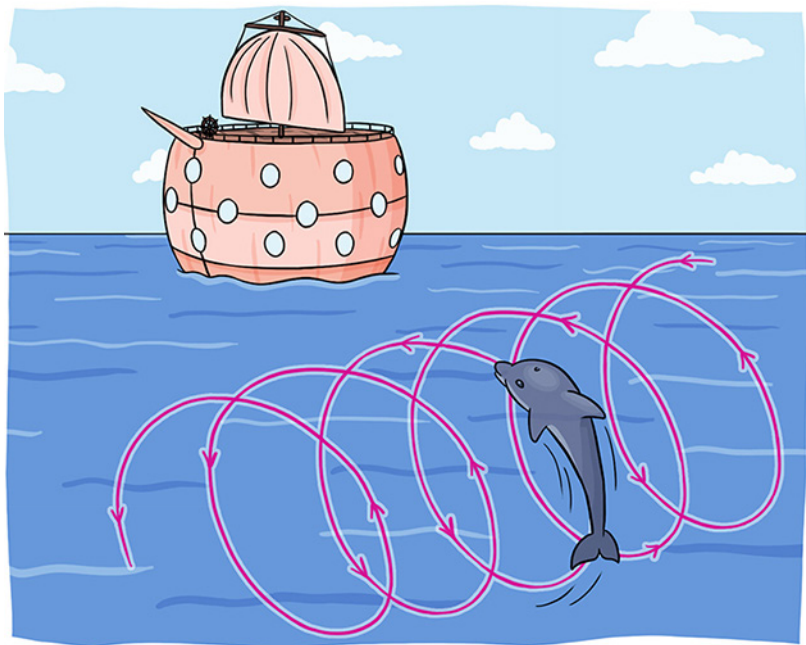
She waits for us to catch up, and again approaches the side of our vessel. I throw her a fish as a reward. In that race, Kaluza limited her movements to just one **dimension**, as did we, moving in a straight line. But Kaluza can move in much more creative ways! I move my outstretched hand in vertical circles. Kaluza sees, and jumps into the air, flipping in a perfect circle. I motion my hand again, and she does the same move but underwater, flipping in a tight tumble turn. I throw her two more fish as a reward.

For her next trick, Kaluza will combine her flipping and her incredible speed in another race. Ready, set, go! This



time, instead of keeping her fin on the surface, Kaluza makes use of the extra dimensions underwater. She spins in a helical path alongside us. Sometimes, we see her fin break the surface; other times, just the glint of her white belly is visible underwater.

This time, though, we sail ahead of her. After a few minutes, she trails behind the vessel, still spinning underwater. We have a clear advantage: While all of our energy is channeled into forward movement, Kaluza is expending some of her energy on the rotation.



It's a trick that our vessel could not pull off. We are far too large to twist in the tight circles that Kaluza can flip in, and I'm afraid you would all drown or fall overboard if I attempted it. Sailing along the surface, we have access to fewer dimensions of movement than Kaluza. In fact, if it

hadn't been for witnessing Kaluza's tricks, we might never have known that such a movement was possible at all.

Kaluza's circle could be called a **compact dimension** because, to a large enough object, it is hidden. Even more dimensions might be hidden from us in similar ways. If Kaluza were a tiny plankton instead of a dolphin, would she be able to move in even stranger ways? What if she were a subatomic particle?

In **Threesville**, we believe that particles move in straight lines through **3D** space. Since particles are so small, though, it's theoretically possible that they also move in compact, hidden dimensions. A particle might take Kaluza's helical path, with a component of movement in the **fourth dimension**. Theoretical particles that move in such a way are called **Kaluza-Klein excitations**. Now you know her namesake!

We saw that Kaluza lost the race when she was swimming in a helix. This is because she had a component of speed in the extra underwater dimensions, slowing down her **linear** progress. Expending this rotational energy is a disadvantage in our race, but it might hold the key to uncovering hidden dimensions.

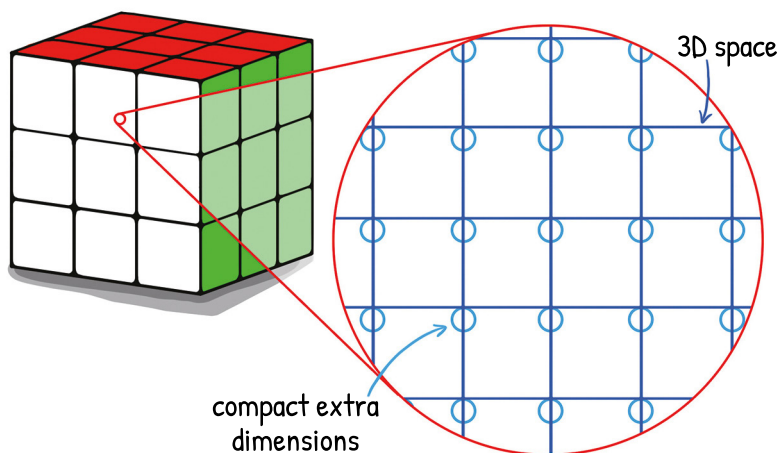
Scientists back home are searching for Kaluza-Klein excitations, and finding them could mean that extra dimensions are more than just a figment of our imagination. A photon—that is, a particle of light—moving along a helical path would move slower than the usual speed of light. A photon that only moved in circles, like Kaluza's circular flipping trick, would appear stationary and, thus, be invisible to us, as its only movement would be in a direction we cannot see (that of **ana-kata**).

Any **4D** movements of Kaluza-Klein excitations are invisible to our detectors. Movement in a hidden dimension

means that the particle has more total energy than we would expect. Finding the total kinetic energy would require knowing the speed of the particle across all dimensions. Through Einstein's famous equation $E = mc^2$, we know that mass and energy are interchangeable. So, it's not just the total energy and speed of the particle that we might get wrong, but its mass, too.

As we've been exploring the **fourth dimension**, I can almost hear the question bouncing around your head: "Does the fourth dimension actually exist?" I have to level with you: As of now, we haven't detected any signs of any higher dimensions, and evidence tilts toward a "no" if you're asking about very large extra dimensions, akin to the ones we've ventured through.

However, compact dimensions like the ones mentioned above do seem to be the best candidates for extra dimensions existing. These compact dimensions would have sizes much smaller than a millimeter. Experiments are done every day, searching for signs of hidden rotational energy or strangely behaving particles that could reveal the presence of compact dimensions.



The compact spiraling paths of Kaluza-Klein excitations are not the only place where scientists are looking for traces of extra dimensions. There are, in fact, many inventive and audacious experiments that have been tried or are currently ongoing.

Let's wave to Kaluza, then head inside, out of the wind. It's time for afternoon tea.

ICING ON THE CAKE

Welcome to our ship's kitchen, where all of our weird and wonderful dishes are prepared. Take an apron hanging from behind the door—things can get a little messy in here as we prepare our own treats.

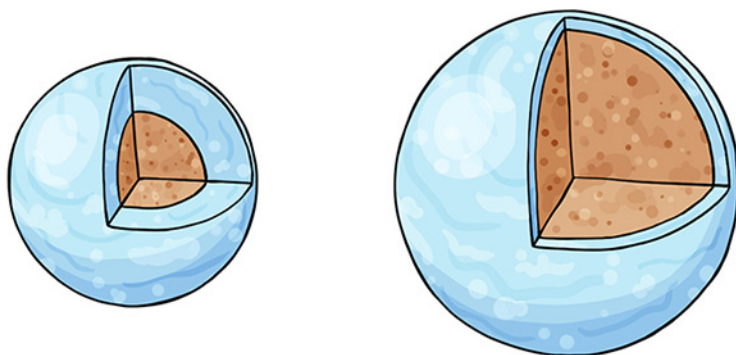
See this tattered cookbook on the bench? It provides the instructions for everything we make. It used to belong to my mother, and she got it from her mother. In fact, this cookbook has been passed down through many generations since 1687. It's called *Newton's Culinary Laws*.

The recipe that we're going to make for afternoon tea is called the Gravitational Globe Cake. It has a solid chocolate core made of green and blue marbled chocolate, causing it to look like a little Earth. Surrounding the core is a spherical sponge cake. The cake is baked inside molds, of which I have several different sizes. Some people prefer a small layer of sponge, whereas other people like their cakes big and fluffy.

On the bench already are some cakes that I prepared earlier. Our afternoon activity will be to apply icing to each of the cakes. Little jars of icing can be found in the fridge. Importantly, the recipe tells us to apply one full jar of icing to each cake, regardless of its size.

The icing is gravity-flavored. When applied to a small cake, the icing is thick, with a very strong taste. When applied to a larger cake, the same jar of icing will be spread in a thinner

layer, since there is more surface area to cover. Some people prefer the more subtle taste when applied like this, but they will consume the same amount of icing overall.



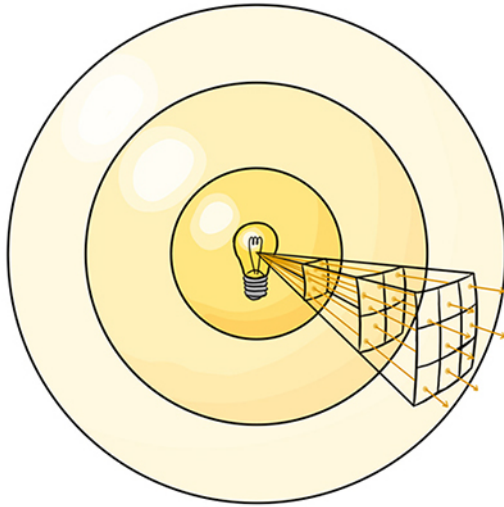
Gravity-flavored icing certainly is an unusual flavor. You might be more familiar with vanilla or strawberry. This icing helps to hold the entire cake together. Without it, the cakes would crumble apart, and I would have nothing but a plate of crumbs to serve to my passengers.

It was Sir Isaac Newton, the author of our cookbook, who explained that the farther you are from the center of an object, the less you feel its gravitational pull. It was over 300 years ago that Newton unveiled this law, and we are still trying to make sense of it today. You can taste this law for yourself by taking a bite out of a few differently sized cakes. The one with its icing layer the furthest from the center has the weakest taste per bite.

Gravity is the force of nature that keeps us grounded. You can think of it like an invisible **sphere** spreading out from Earth. Gravity's strength is tied to the surface area of this invisible sphere. The farther the sphere extends away from Earth, the more space it covers, and the weaker gravity's pull.

Gravity isn't the only thing that acts in this way. We encounter a similar "spreading out" of phenomena like light

and sound waves every day. A lightbulb appears blindingly bright when it's close to your face, but when seen from far away, the lightbulb appears dimmer, since its light is spread out over a large area, and you are only seeing a tiny part of it.

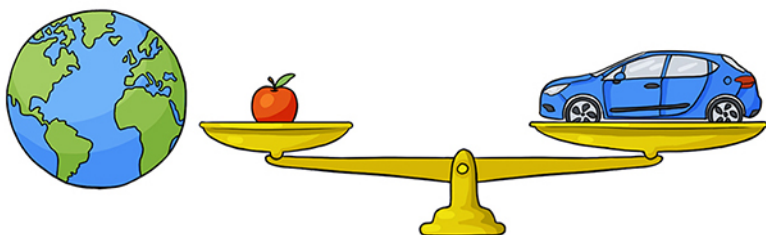


The surface area of a sphere is determined by the relationship $4\pi r^2$, where r is the distance from the center of the sphere. This relationship helps us understand exactly how the strength of gravity, and the taste of our cakes, changes as we get farther away from an object like Earth.

If we increase the radius of a spherical cake, its surface area increases in proportion to r^2 . The same amount of icing, now spread over a larger area, becomes thinner—its strength at any point drops by $1/r^2$. This is the inverse-square law. It describes the icing on our cake, but it also describes gravity, light, sound, and even the **electromagnetic force**.

Let's look at an example regarding gravity. A car normally weighs about 1000 kg on Earth's surface. If we could place the car at a distance that's 100 times Earth's **radius**, the gravitational pull would decrease by a factor

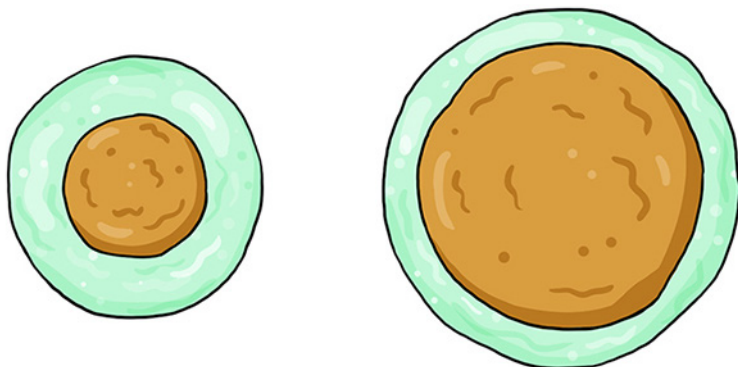
of 10,000 (since $100^2 = 10,000$), and the car would weigh only 100 grams, the usual weight of a small apple on Earth.



How would you expect this relationship to change if we lived in a world with a different number of dimensions?

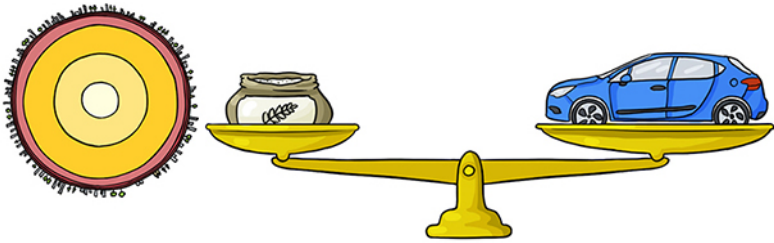
In our cookbook, we can find a recipe for cookies. Baking these thin, flat cookies is a way to taste how things would work in a **2D** world. We apply icing just around the edge of each cookie, seeing that the icing spreads out over a circle instead of a sphere.

Now, a cookie with twice the radius of its neighbor will have icing half as thick. The circumference of a circle is found by the relationship $2\pi r$, where r is the radius. Since it scales with a single r —not r^2 —the icing thins more gradually than it did with the spherical cakes.



In a 2D universe, the strength of gravity wouldn't follow the inverse-square law, but instead drop by $1/r$. A car placed

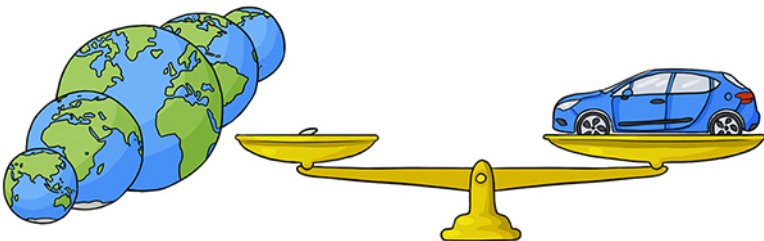
100 times farther from the center of a flat, 2D Earth would experience a gravitational pull that's 100 times weaker. Its weight would decrease from the equivalent of 1000 kg to just 10 kg—the same as a large bag of flour on Earth.



There's one more recipe that I would like you to try. This one's for **Globe** Cakes. When trying to cover the **hypersurface** of a four-dimensional cake, we run out of icing much sooner. The **hypersphere** is even more spacious than the sphere, and we fail to coat the entire hypersurface with our standard jar of icing.

The relationship that can be used to find the hypersurface 'area' of a glome is $2\pi^2 r^3$. This is actually a 3D volume. We now have a factor of $1/r^3$ linking the radius of the cake to the strength of the icing flavor. The thickness of the icing drops off steeper than ever before, so we will now have to make smaller cakes if we want to have any hope of covering them in icing.

If gravity behaves likewise in a 4D universe, then a car placed 100 times farther from the center of a hyperspherical Earth would experience a gravitational pull that's 1,000,000



times weaker (since $100^3 = 1,000,000$). At that distance, our car would weigh just 1 gram, the same as a cooked cannellini bean on Earth.

If we ever measured gravity behaving in this way, it might hint at the existence of a fourth dimension of space. However, so far, all our measurements of gravity, including those pondered by Newton, match the 3D model. Gravity in our world decreases at a rate proportional to $1/r^2$, a rate that fits with a 3D space—not a 2D or 4D one.

However, don't lose hope, 4D enthusiasts! There's a tiny crack in the door of possibility. We haven't been able to measure gravity super up-close, at distances of less than a tenth of a millimeter. So, who knows? There might be a hidden 4D world where gravity plays by different rules. Scientists are on the case. They're constantly making experiments more sensitive to peek into these uncharted territories. If they find gravity behaving differently at these tiny scales, it could be time for a new recipe.

THEORY OF EVERYTHING

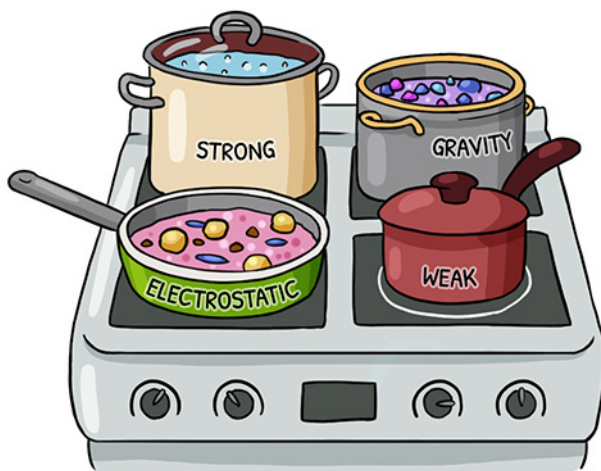
Standing by the stovetop at the back of the kitchen is the ship's cook: Say hello to Toebee! She is working on an experimental dish at the moment—let's go and take a look at it.

Scattered all around Toebee are soybean pods. Soybeans are such a remarkably versatile ingredient that they can be used to make what we call here a one-ingredient soup. The soup contains four distinct elements: edamame, tofu, soy milk, and miso paste. These four elements may seem different from one another, but they are actually all made by preparing soybeans using different methods.

Edamame is made by boiling immature soybean pods. Tofu is made from dried soybeans that are soaked in

water, crushed, and boiled. Soy milk is made by soaking and grinding the beans. Finally, miso paste is made from fermented soybeans. So this delicious soup is really just soybeans shown off in four different forms.

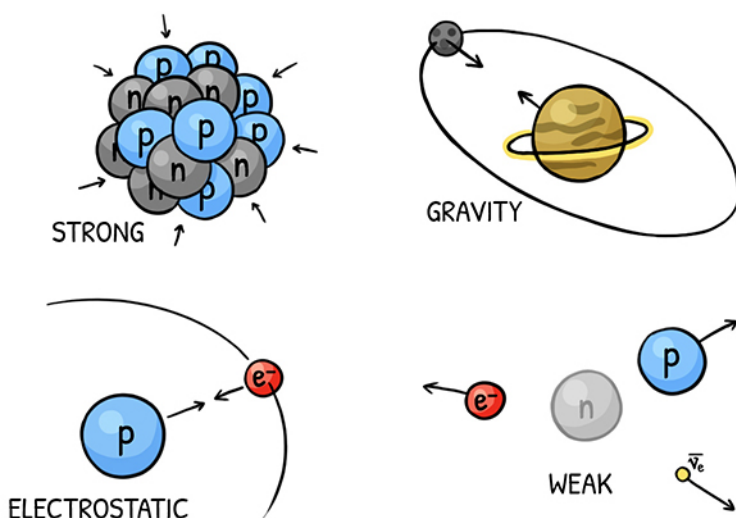
You are right to wonder what on earth soybeans have to do with anything. Well, our universe also appears to feature four distinct ingredients, or fundamental forces: the electromagnetic force, the **weak nuclear force**, the **strong nuclear force**, and the **gravitational force**. Toebee is trying to find out if these forces, like soy products, might actually be manifestations of the same thing. If there was one fundamental force, then could our universe also be described as a one-ingredient soup?



Knowing that soybeans can be used to make many different products allows chefs to make a plethora of interesting foods. Scientists would be really excited if our own universe turned out to be a one-ingredient soup, because it could allow us to use the laws of physics to their full potential. It would, at minimum, be very intellectually satisfying to boil down the universe to one underlying

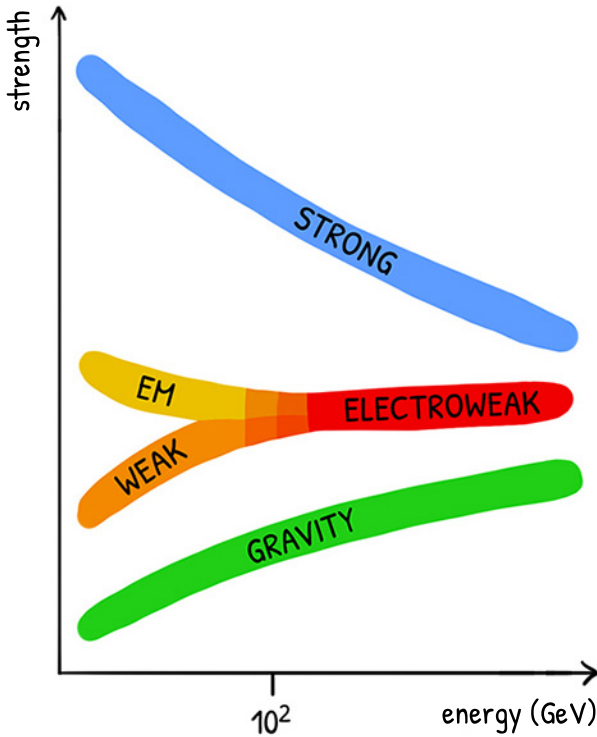
fundamental force. It would offer us the slightest feeling of understanding what our universe really is.

In Toebee's stovetop experiment, she has pots boiling each of the fundamental forces. The electromagnetic force deals with charged particles, like the invisible tie keeping electrons circling the nucleus. The weak nuclear force is a shy character that oversees radioactive decay and plays a crucial role in nuclear reactions such as fueling the Sun. The strong nuclear force is the powerhouse, holding the nucleus together against the repelling power of protons. Lastly, the gravitational force, or gravity, is the force of attraction between masses; it keeps planets in orbit and your feet firmly on the ground.



Toebee's experimental results so far are quite intriguing. She has found that some of the forces do appear to change when enough energy is applied to them. She cranks the dial on the stove, applying more energy to the pots, and finds that the electromagnetic and weak nuclear forces both turn into the same thing. She has nicknamed this mushy

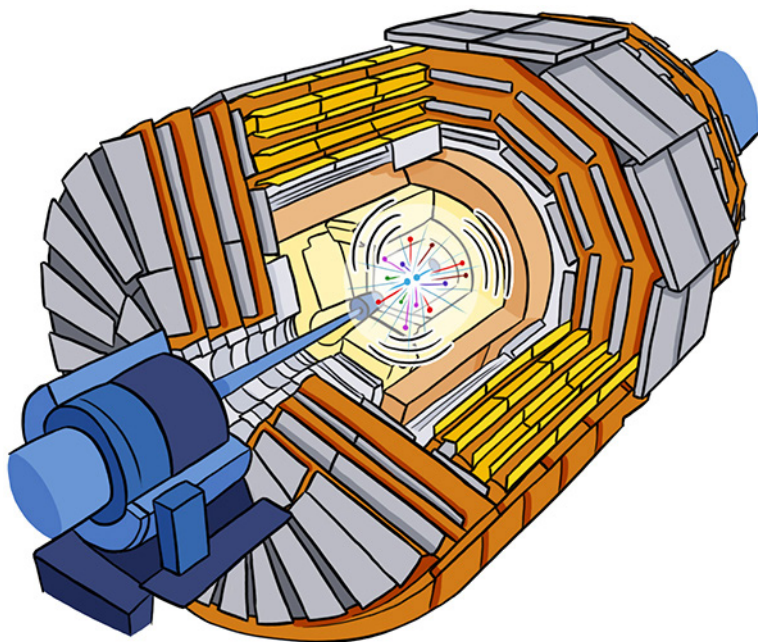
result the **electroweak force**. Tasting a spoonful of the electromagnetic and weak nuclear pots confirms they are now indistinguishable.



The pot containing the strong nuclear force is the next one to start changing. Toebee cranks the dial even higher, applying a great deal of energy to study the force. The strong nuclear force starts to look a bit like the electroweak mush. She tastes it and can tell it's nearly the same. Her stove dial doesn't go any higher, but Toebee knows that the strong nuclear force, with a bit more energy, will become indistinguishable from the electroweak force. She's now pretty confident that these three forces might really be different flavors of the same thing.

What troubles Toebee is her gravity pot. Despite all the energy she's put into it, it seems barely affected. Unlike the other three forces, which seem to be converging, her gravity pot stubbornly remains its own distinct thing. Gravity seems to be letting down our dream of making a one-ingredient universe soup.

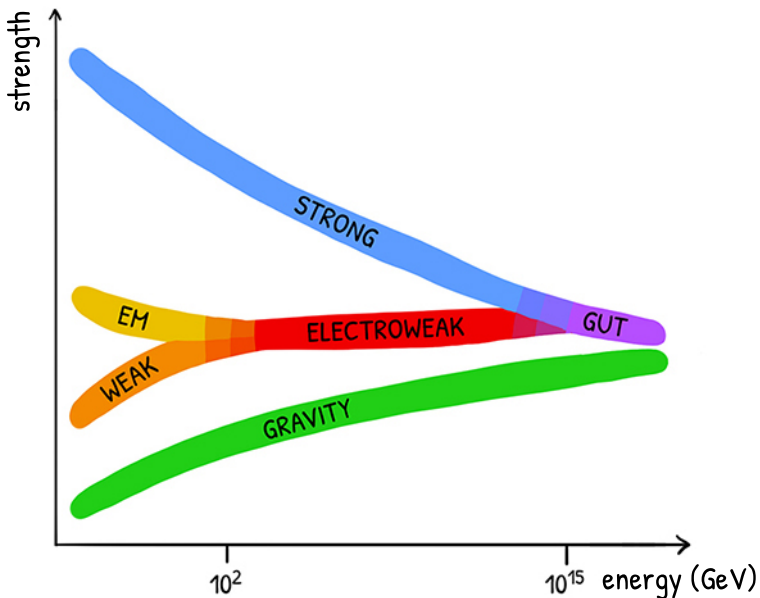
Toebee's stovetop experiments aren't just abstract ideas; they capture some of the most exciting ideas being pursued in physics. The combination of the electromagnetic and weak nuclear forces into the electroweak force has been experimentally established. Instead of standing at a stovetop, particle physicists found the evidence they needed using high-energy particle accelerators.



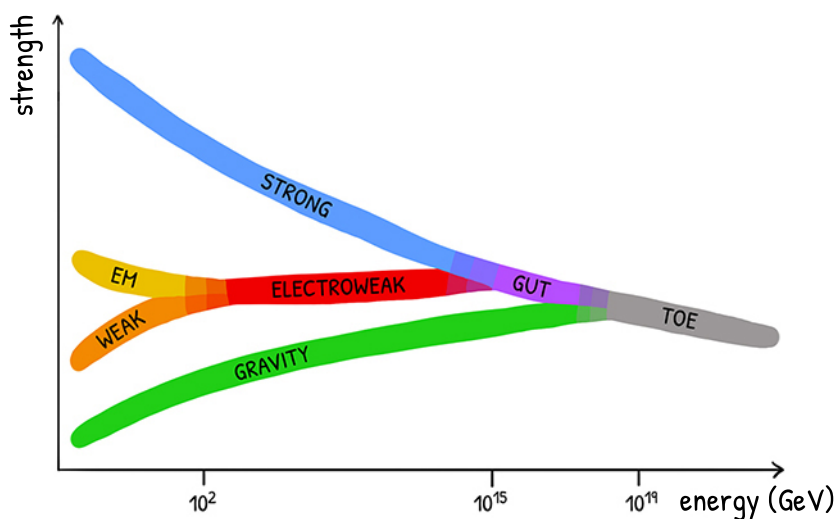
Colliding particles at extremely high energy using accelerators allows these forces to be studied. Increasing the energy of the particles being studied means we can

probe how forces act over smaller distances. It's the smallest distances that remain a mystery to us, until we find a way to increase the energy we are applying to them.

Even though our particle accelerators are not currently powerful enough to experimentally offer proof, scientists have been able to theorize what might happen at higher energies by extrapolating the trends in the data beyond the region we have measured. The point at which the strong nuclear force is expected to become indistinguishable from the electroweak force is very exciting to think about. Work done on understanding this concept falls under the name of **Grand Unification Theory (GUT)**. At that level of energy, electromagnetic, weak nuclear, and strong nuclear forces would all become equally powerful. At a fixed distance away from a particle, all three of these forces would have the same strength and thus be indistinguishable from each other. So it would appear as if just one force is acting on the particle.



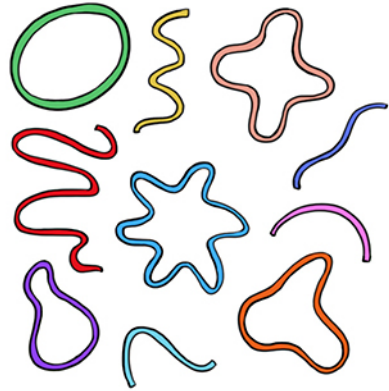
This unification party sounds exciting, but unfortunately, gravity is late to the party. Gravity's path to unification appears to be much further out. It isn't until much higher energy is applied that we can hope to find a point where gravity is the same strength as the three other forces. At this point, all four fundamental forces would be indistinguishable from each other and treated as one. A theory that could describe this one force is the elusive **Theory of Everything (TOE)**.



TOE is what Toebee is hoping to find hints of at her stovetop. If it exists, then our universe could be considered a one-ingredient soup, with one fundamental force manifesting in different ways to create everything that we know. Toebee isn't the first to look for TOE; many of the greatest scientists throughout history have spent years of their lives looking for it, but so far it remains a mystery. Its existence is one of the biggest unanswered questions in physics.

If we might ever be so lucky as to find TOE, then what is that one ingredient that makes up our soup? What is our "soybean"? One idea is that the ingredient may be tiny

vibrating strings. This idea is from **string theory**, which postulates that everything is made of strings, vibrating in many compact extra dimensions. Each vibration pattern of these strings could correspond to different particles and forces observed in our universe.



The size of these extra dimensions predicted by string theory is a topic of great intrigue. Some think that they could be on the scale of the Planck length, an almost unimaginably small scale of about 10^{-35} meters. However, other versions of the theory suggest the existence of larger (yet still tiny) extra dimensions that are a fraction of a millimeter in size.

Generally, the smaller the extra dimension, the higher the energy required to probe it. This presents a challenge in testing string theory, as we currently lack the technological capability to conduct experiments at the energy levels where the effects of these vibrating strings might become apparent.

One question remains, though: Why is gravity so fashionably late to the party? Watch Toebee again as she experiments with the forces. Notice her lifting the gravity pot and examining it from all angles. She is checking to see if there are any leaks in the pot. If gravity is “leaking” out, then maybe that could explain why it is comparatively weak compared to the other forces acting over the same distance. But where could gravity be leaking to? Well, perhaps a fourth dimension of space!

Remember the game of lawn bowls I told you about in Part 2, where my mouse friend Susan was stumped by energy escaping into a third dimension? We too might

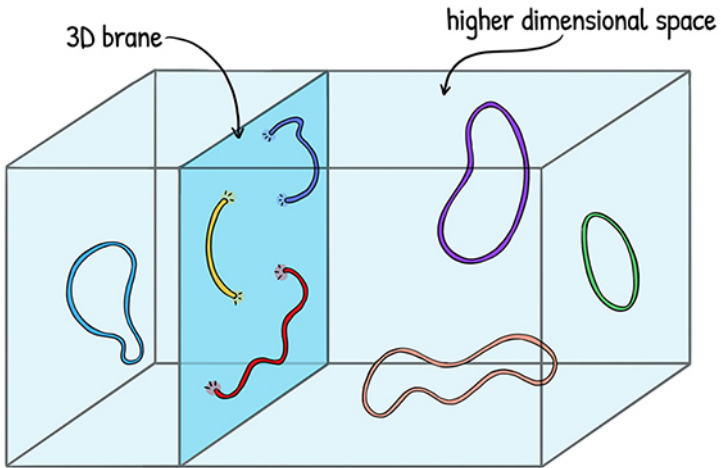
miss energy if it escapes to a higher dimension. For example, take the **graviton**, the particle thought to carry the gravitational force. If gravitons move into a fourth or even higher dimension of space, then the gravitational force observed in Threesville will be diluted.

If we could wrangle the gravitons to stay in Threesville, then I wonder if gravity would unify with the other forces at a low enough energy for us to study? This theory about gravity's 4D escapade is not considered a probable explanation, rather just one of the more "out there" ideas. It does bring an additional question along with it, though: If gravitons can escape into a fourth dimension, then why don't the other force-carrying particles do the same? The electromagnetic, weak nuclear, and strong nuclear forces seem to be confined to just three dimensions, so it doesn't seem fair that gravity would be given a bigger playground.

String theory, again, has some ideas for ways to explain this. In string theory, **branes** are the underlying structures of reality. Branes can have any number of dimensions. String theorists might have you believe that our universe is a 3D brane that is embedded in a higher-dimensional space of 10 or 11 dimensions. Other branes could be extremely close to ours but separated in the direction of a higher dimension that is inaccessible to us.

The vibrating strings of string theory could be closed loops or open, like a cut loop with two exposed ends. These ends of open strings might be attached to branes. Each end attached to a brane could give rise to a particle that exists there.

Because open strings are firmly attached to specific branes, they are confined to the dimensions of the brane to which they are bound. Fundamental particles such as the photon and those giving rise to the electromagnetic, weak



nuclear, and strong nuclear forces might be confined to our 3D brane because they are from open strings. Since we are made of these particles, we are also theoretically trapped on our 3D brane.

Gravity, however, could be a closed string. This uncut loop could bounce around between branes, not bound to any particular one. We might just experience the ripples of gravity as it is mediated by higher dimensions. This idea is tempting because it could explain why gravitons have been so elusive to detect so far. However, string theory introduces its own problems, and without claims that can be tested experimentally, it will fall out of favor and be replaced by theories that offer a means of evidence.

While our current particle accelerators and other experiments are not yet operating at high enough energies to say anything about string theory or to spot compact higher dimensions, the work done by experimentalists has helped to place bounds on how large extra dimensions might be if they do exist. Our current upper limit looks to be about a tenth of a millimeter. That might mean that **Foursville's**

Hintonland is a possible tourism destination for only the tiniest of travelers.

Toebee will remain at the stovetop, working on her one-ingredient soup for a while longer. Will she be able to boil our universe down to one fundamental force explained by a Theory of Everything? Only time will tell, but we wish her luck.

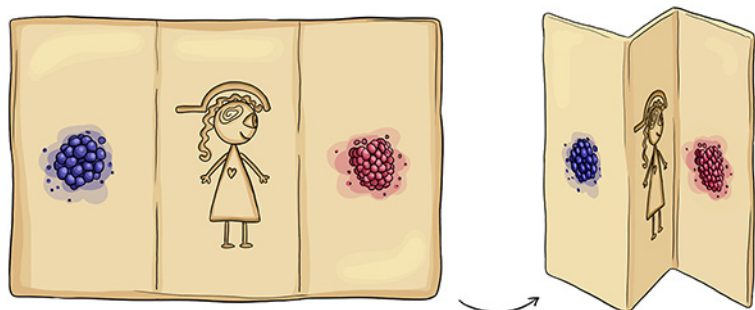
Lemon jumps up gracefully onto the kitchen countertop, a place I've told her many times she is not allowed to go. Before I can catch her, she walks over to the next bench I wanted to show you: our pastry station. Lemon walks through a dusting of flour and leaves paw prints behind her in a trail. The oval-shaped toebeans on the bottom of her feet leave an especially cute impression.

FOLDING PASTRIES

By twisting and folding dough, we can play with what it might mean for us if gravity really is leaking into a fourth dimension. We can construct some of the far-fetched yet geometrically plausible ways that such an idea could manifest in physical phenomena.

On the bench before us, you will see many dough sheets rolled out flat. The rolling pin has squashed them into approximately 2D sheets, waiting to be twisted, folded, and baked into delicious 3D pastries. Try to play with the sheets in your mind: What shapes can you imagine making? Can you imagine doing the same with 3D dough, folding it into a 4D pastry?

Any manipulation of a 2D sheet can be used as an analogy for 3D space. With a skewer, we etch the outline of Tudy, whom we met in **Twosville**, into one of the flat sheets of dough. On either side of her, we squash a big juicy berry into the dough. Now fold the sheet up like a brochure, with two creases. Let's put it in the oven for a little while and see what happens.



Out of the oven, we get a tasty pastry that began its life as a 2D structure but has been folded through the third dimension to make something more interesting. Tudy, wedged in the middle, is a 2D creature confined to the sheet she was etched on. This pastry is her whole universe. The berries might represent distant galaxies in front of and behind her. She would have to travel along the entire sheet to reach either of them, even if in 3D space they are quite close. She can't jump up or down to cross the shortcut gap.

The juices from each berry have seeped out into the pastry, forming two stained regions. You could consider these juices to be the light from each galaxy. The juices are confined to exist within the dough. We believe light is not able to travel between dimensions and, in this case, would also be confined to two dimensions. Thus, the galaxies would appear far away from Tudy, even if there is a shorter path through 3D that exists.

In addition to light, each galaxy also exerts a gravitational force. This might be represented by the smell of the berries, which permeate not just the dough but the three-dimensional space through which the pastry is folded. Smell, like a gravitational pull, might spread out in all directions, just like an expanding sphere. Tudy would be able to smell the galactic berries oriented above and below her. Therefore,

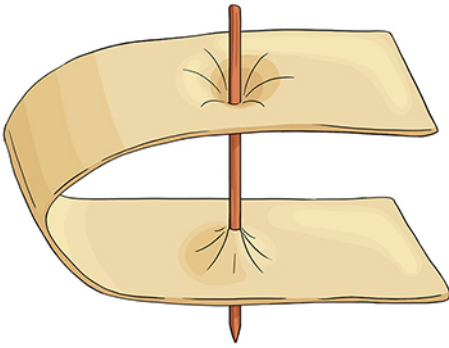
she might be confused about why the berries smell so close, yet their light looks so far away. The two signals contradict each other.

If our universe were similarly folded into a four-dimensional pastry, perhaps we would also observe weird gravitational behavior. Playing with pastries as analogies for our universe makes us wonder if mysteries such as **dark matter** might have something to do with gravity extending into higher dimensions. If light remained confined to our usual three dimensions, then we could potentially feel the gravitational effects of a distant galaxy, perhaps billions of years before we were actually able to see it.

Dark matter is the name we give to the seemingly invisible mass that pulls together distant stars and galaxies, moving them in ways unexplained by the visible universe alone. It's akin to hearing the rustling of leaves and knowing there's something unseen that stirs them.

Until we find more evidence of what this dark matter is, many different ideas rustle around. These possible explanations include new particles we haven't detected yet, an altered law of gravity, and even billions of small black holes. Experiments are currently ongoing all over the world as scientists look to solve the puzzle of dark matter. These experiments may lead to an explanation that doesn't involve higher dimensions, but it remains an intriguing possibility.

There are other baked goods coming out of the oven now, including croissants, doughnuts, and pinwheels. Any of these shapes could be interpreted as something geometrically possible in higher dimensions. Next to our berry brochure is another dough sheet that has been folded in half and stabbed with a skewer. This is one interpretation of a **wormhole**—a shortcut between two regions of space. Imagine 2D inhabitants of the sheet crawling up the surface



of the skewer to the top layer.

Shortcuts through space are of great interest to us. Humans love the idea of being able to set sail for interstellar space and visit other stars,

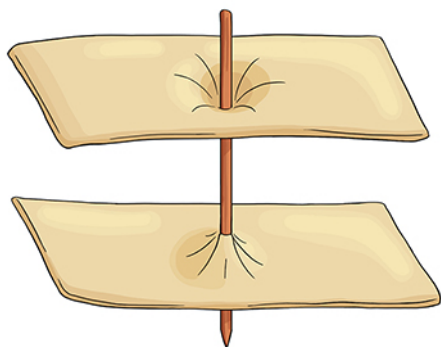
exoplanets, and galaxies beyond our own. It appeals to our sense of adventure, but may also be what's needed one day in the distant future in order to ensure our survival as a species. The Sun that we orbit is about halfway through its lifecycle and has about 5 billion years left on the clock before it expands into a red giant, taking Earth out with it.

Before that happens, we need to find a second home, Earth 2. There are plenty of candidates among the exoplanets that orbit other stars. The problem is that the closest star, Proxima Centauri, is four light years away and will take at least 10,000 years for us to travel to, given current efforts in space flight. If the idea of the wormhole could extend to our space, then a wormhole might provide a shortcut to the stars.

The drawing above is often used to illustrate wormholes, but remember that's what they would look like for 2D creatures but not for us. Drawings like this are a happy accident born out of the need for simplicity. Raise this up a dimension and imagine folding our 3D space over through a fourth dimension, just like the pastry folded in the third dimension. Instead of circular holes at the wormhole's entry and exit, there would be spheres in our world. Instead of a three-dimensional tube connecting them, for us it would be a four-dimensional tube.

Wormholes excite us as shortcuts through space, but in a special case they could also be portals through time. To see why, you need to know just a little bit about black holes and Einstein's work on relativity. Einstein essentially was able to show, through his equations, that mass has an impact on the space and time around it and a very large mass could distort the time and space nearby to such an extent that time would slow down.

A black hole is an intense concentration of mass in a relatively small area, and it distorts the space and time around itself so much that time slows to a stop at the horizon of the black hole. If we got lucky and found a wormhole that had its other end near a black hole, then traveling through it would spit us out in the past, in a region where time runs slower than back home.



Going in one end of a wormhole and out the other might connect two regions of space or two entirely different universes. Imagine slicing the folded pastry along the crease. We would then have just two sheets of dough

connected by a skewer, the start of a "cosmic lasagna."

Are there other "sheets" of 3D space? Some physicists think so. Theories involving branes, as we have already met in string theory, consider the possibility of many universes just like ours existing parallel to each other, stacked through a fourth or higher dimension. We can't contact those parallel universes, but there are some experiments we have tried to look for them. These experiments include looking for traces

in the background radiation of space to see if we could have collided and merged with another universe at some point in our history.

A wormhole connecting two separate universes might be our only way to visit friends in parallel universes. If we looked at one entry of a wormhole, we would see all the light from the universe on the other end funneled through the sphere-shaped opening, and it would look like those mirrored Christmas ornaments that reflect back the room around them, except our wormhole opening would be showing a window into a different room in a different universe.

A very clever version of our future selves might even be able to tie one end of a wormhole to a spaceship and take it wherever they wanted, but right now we can't manipulate spacetime in this way, and the idea of wormholes is shaky as it is. Scientists believe that, even if wormholes existed, they would struggle to stay "open" because, after a short time, the middle of the wormhole would collapse under its own gravity. It would need some force pushing outward from inside the wormhole to counteract that force.

The wormhole pastry is only one way to create a bridge between universes, but the possibilities are only limited by your imagination and what can be done with a sheet of space. Just like inventing a new delicious pastry, math is all about being creative.

Use some trial and error and play around with these ideas yourself. You may feel like the author of your own sci-fi novel. These kinds of ideas toe the line between science and fiction, where we get to imagine the consequences of known physical and mathematical laws. We just need to remember that many of them are currently underbaked and need more work and more evidence before they are ready to come out of the oven.

INFINITELY MANY DIMENSIONS

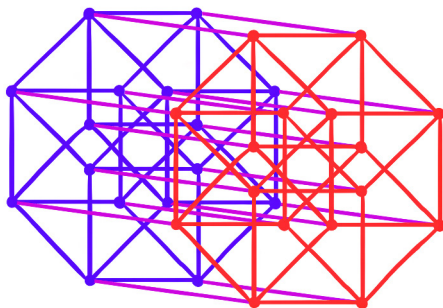
We've spent a lot of time together recently trying to imagine a fourth spatial dimension. You may feel exhausted from all the effort. Depending on your perspective, my next sentence will either bring you joy or sorrow.

In mathematics, at least, the dimensions don't stop at four. We could next move to imagining the fifth, sixth, and seventh dimensions. Every time, we repeat the analogies we formed by looking down at the second dimension. After the seventh dimension, we could keep going all the way to infinity.

"Double it and give it to the next person" refers to a social experiment in which strangers in a public setting are asked if they want to take a dollar or double it and give it to the next person. It is a meme but also a useful framework for us to visualize higher dimensions. As we saw earlier in this book, when drawing shapes of any dimensionality, we stack copies of the shape one dimension lower. Like how dots became a line, lines became a square, squares became a cube, and cubes became a **tesseract**.

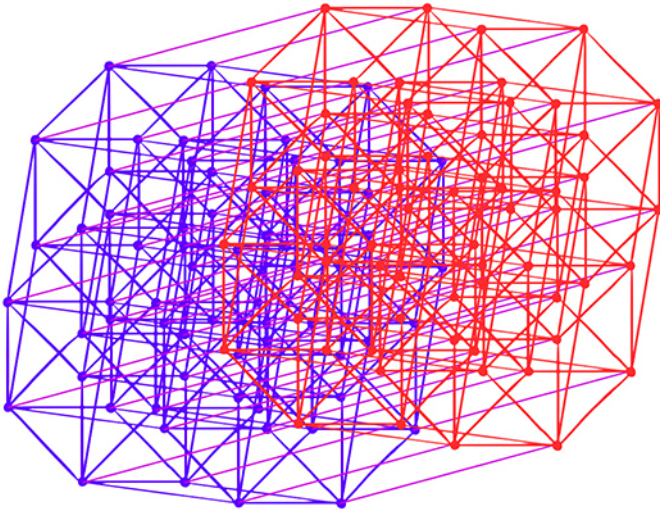
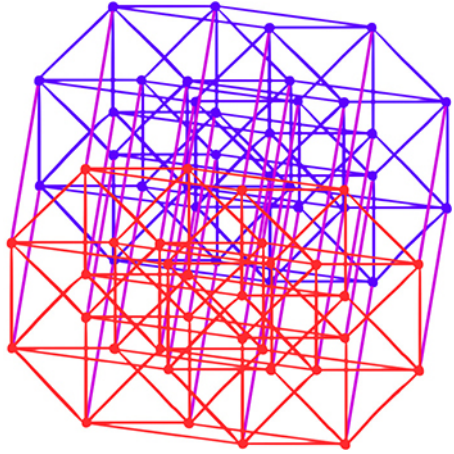
Is it possible to draw higher shapes, including a five-dimensional **hypercube**? You bet it is, as long as you remember how to draw a four-dimensional cube. Just double it and give it to the next dimension.

Take our **wireframe** drawing of the tesseract from earlier and copy it, shifted diagonally across the page. Then, just connect up all the vertices. We will end up with something like this:



To draw a six-dimensional cube, copy the previous image, shift it across the page, and then connect up all the dots.

With the same technique, there is no need to ever stop drawing hypercubes. Here is a seven-dimensional cube:



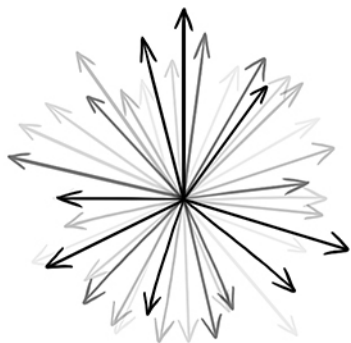
These wireframe drawings are just one way to represent these higher-dimensional shapes on a flat piece of paper. They don't show us the whole picture but go some way to revealing the structure of each shape.

As explorers of the abstract realm, we naturally ask: Can the concept of dimensions be expanded indefinitely? In the world of mathematics, the answer is a resounding "yes." We

just saw one technique to keep drawing hypercubes beyond our wildest imagination, and with simple math we can also describe the **hypervolume** contained by such hypercubes.

A 3D cube contains a volume of L^3 where L is the length of one side. In the same way, a tesseract's volume is given by L^4 . In higher dimensions, we find volumes of L^5 , L^6 and even L^{1000} .

Some spaces have infinitely many dimensions. One such space is called **Hilbert space**. This isn't just a quirky mathematical playground; Hilbert space forms the bedrock of quantum mechanics, a core branch of physics that studies the behavior of the tiniest particles. Within the bounds of Hilbert space, particles aren't just defined by simple factors like their position or speed. Instead, they're represented by wavefunctions, which capture all the strange and wondrous behaviors unique to the quantum world.



In Hilbert space, each possible configuration of a particle—its position, velocity, spin—corresponds to a dimension. Thus, a particle living in a quantum world doesn't inhabit the cozy confines of three-dimensional space, but rather a Hilbert space of infinite dimensions.

This might sound like an extreme case of dimension overload, but then again, quantum mechanics does have a reputation for defying our everyday intuitions.

When a measurement is taken of a quantum particle, the infinite possibilities of Hilbert space collapse to show a particle at a definite position. Having math that can describe an infinite dimensional space allows scientists to calculate the probabilities of how particles will behave.

Hilbert space isn't just useful for understanding quantum mechanics, it also finds applications in areas like signal processing and data compression, among others. This infinite-dimensional space, indeed, plays a crucial role in our understanding and representation of the physical world.

So, yes, our adventure doesn't end in the third, fourth, or even the tenth dimension. It carries us beyond, into the realm of the infinite, to a mathematical landscape where every possibility exists.

A SOUR FIND

The mystery of the stolen gems lingers like a dark cloud upon our vessel. Not a whisper of the whereabouts of our own gems, and those belonging to the beings of the 2D and 4D realms, has graced our ears. A chilling thought seizes me: Could the thief be a visitor from the dimensions beyond the fourth? Every notch higher in the dimensional ladder means a more complex and elusive suspect. The prospect of venturing into the fifth dimension aboard this ship sends a shiver down my spine. Unpredictable spatial effects could spiral us into uncharted waters, where we might struggle to keep our heads above water.

If the gems are not found soon, chaos might erupt in our mathematical universe. The 2D beings, in their panic, might encroach upon the **1D** world, claiming what isn't theirs. More alarming yet, the 4D entities might breach the walls of our 3D space, their intentions and capabilities unknown.

We, the interdimensional travelers, could be caught in the crossfire, bearers of a peculiar burden to restore the peace. To prevent all-out chaos, a plan takes shape in my mind: a decoy, a false gem with enough allure to draw the thief into the light. I cast a glance at you, our resident intellect, as I will need your help.

We spot a scrap of aluminum foil left on the pastry-making bench in the kitchen, its metallic sheen holding promise. We could use this as bait. Please, if you don't mind, dash to the theater and grab the torch from the projector!

Perfect, thank you! I've scrunched the metallic foil into a ball. The once 2D sheet now folds in on itself with detail at every scale, like a **fractal**. If you shine the torch onto it and rotate it, you might create reflections from the irregular surface, scattering the light in all directions, which will be irresistible to our thief. The aim is to mimic the glistening of a gemstone.

Amid our ingenuity, Lemon, my beloved cat, eyes the foil with mischief, unfazed by the gravity of our mission. A playful snatch, and our false gem is held hostage by tiny, innocent teeth. A laugh almost escapes my mouth, but the levity of the moment is cut short by an accusation—from you, of all people!

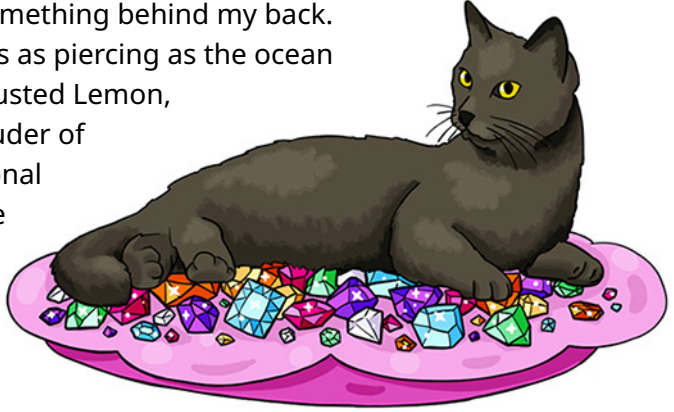
No, it is not possible that Lemon, my trusty companion, could be the thief! I am offended that you even asked. She must have smelt the leftover crumbs of pastry dough on the foil.

To prove her innocence, let's tread the path downstairs to Lemon's favorite corner. She has beaten us to it, and is sitting on her pillow with the metallic foil still in her teeth. Maybe we should try again with another object; I could shatter some glass to make a new false gem.

But as we watch Lemon together, I'm sure you see what I see: Lemon tucking the metallic ball beneath her pillow and initiating a series of tinkling sounds. Hundreds of small, stolen gems shift under her weight. Here, amidst the quiet purring and the gentle lapping of waves against our ship, a revelation stuns us both.

With every hue of the rainbow reflected in my wide eyes, I must confront the fact that my constant companion has

been up to something behind my back. The betrayal is as piercing as the ocean breeze: my trusted Lemon, a silent marauder of interdimensional treasures. The thief had been aboard all along, hiding in plain sight.

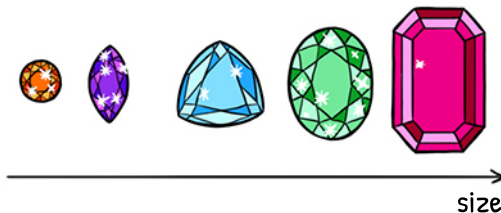


DATA SETS

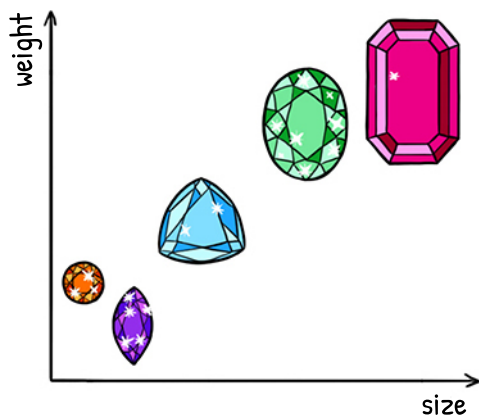
I am embarrassed that you had to bear witness to this fiasco. Please assist me in retrieving the gems from their warm hiding spot. We will want to lay them out on the table to catalog what we have and begin the task of returning them to their rightful owners. In my shame, I can at least offer you one more secret here as we sift through gems.

Higher dimensions aren't just mind games; they are useful for many things. That's because dimensions aren't just about space or even time. In fact, dimensions can be whatever you want them to be. A dimension can be any measurable quality.

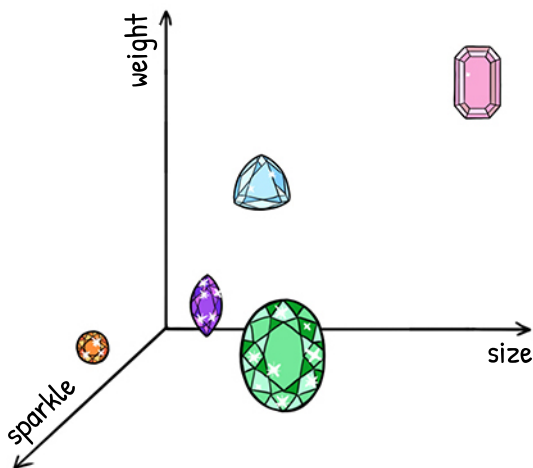
We now have a pile of gems of all different kinds. Let's arrange them by a common quality. One measurable quality of a gem is its size. Let's position each along a straight line based on their size, from smallest to largest. This is a one-dimensional arrangement.



Now, let's consider another quality: weight. We can then map the gems in a two-dimensional plane. Every point within this plane represents a unique combination of size and weight. Large, heavy gems are in the top-right corner, while small, light gems occupy the lower-left corner.



Gems have more than two qualities that define them. Another quality worth considering is the intensity of the gem's sparkle. This characteristic of the gem is something we also want to keep track of. We will do this by adding another dimension to our arrangement: the dimension of sparkle.



The dullest gems of each size and weight will stay where they are, resting on the table, but those with intensity will be stacked on top. Gems with the most sparkle will be placed on the very top of the now-very-precarious block of gems. We have a three-dimensional space where each gem is a point defined by its size, weight, and sparkle.

But a gem is not just defined by its physical attributes; it holds within its crystalline structure a story of rarity. How do we plot this, when every corner of our three-dimensional space is already claimed by the gem's size, weight, and sparkle?

Enter the realm of higher dimensions, a space where the complexity of our gems can be fully expressed, where each attribute, from sparkle to weight, size, and rarity, can be plotted, visualized, and understood. To represent this fourth quality in a manner similar to the first three, we'd require an additional spatial dimension. So, the idea of a four-dimensional space can be imagined as a realm where points can represent four distinct qualities, just as our familiar three-dimensional space represents three qualities.

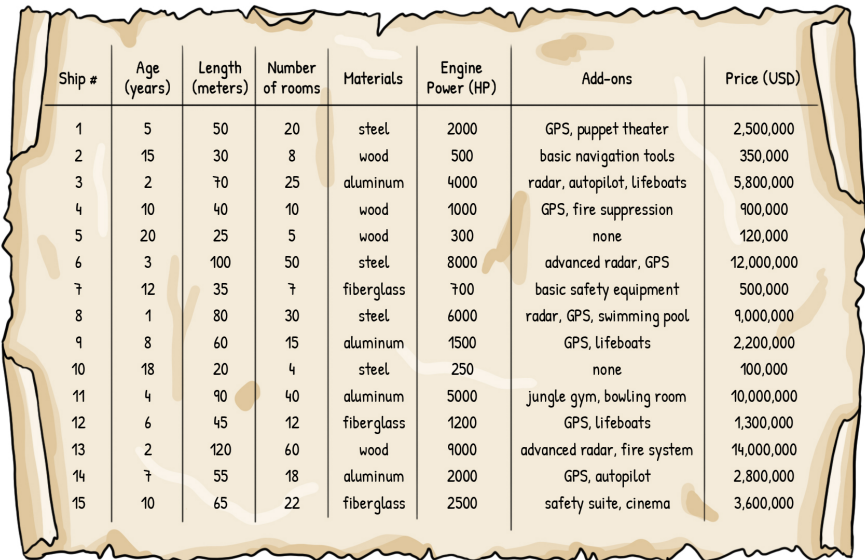
Can you think of any other characteristics of the gems? Maybe their origin, color, or smoothness. These would add fifth, sixth and seventh dimensions to our grid of gems. For a gem to be totally cataloged may take upward of 10 dimensions. Fully describing the gems within our catalog will map out **hyperspaces** of many dimensions. We don't need to be able to imagine what these hyperspaces will look like; it is enough to know how to describe them mathematically in order for them to be useful.

When we're looking at something as complex as a data set, the term **dimension** starts to take on a different meaning. In a data set, each data point might have several pieces of information associated with it. These pieces of

information are referred to as **features**, **variables**, or **dimensions**.

Multi-dimensional data sets are everywhere. They could be information collected about customers in a store, types of products, or even physical attributes of materials. People from all walks of life are spending their time thinking about hyperspace without even knowing it.

Let's take the example of predicting ship prices. The price of a ship is not just determined by one single factor. There are many variables that contribute to the price of a ship, such as its age, its length, or the number of rooms it has. The materials, too, play their part. And let's not forget the add-ons. Perhaps a jungle gym for dimensional games or an onboard puppet theater could add value. Each of these variables is a dimension in our data set.



Ship #	Age (years)	Length (meters)	Number of rooms	Materials	Engine Power (HP)	Add-ons	Price (USD)
1	5	50	20	steel	2000	GPS, puppet theater	2,500,000
2	15	30	8	wood	500	basic navigation tools	350,000
3	2	70	25	aluminum	4000	radar, autopilot, lifeboats	5,800,000
4	10	40	10	wood	1000	GPS, fire suppression	900,000
5	20	25	5	wood	300	none	120,000
6	3	100	50	steel	8000	advanced radar, GPS	12,000,000
7	12	35	7	fiberglass	700	basic safety equipment	500,000
8	1	80	30	steel	6000	radar, GPS, swimming pool	9,000,000
9	8	60	15	aluminum	1500	GPS, lifeboats	2,200,000
10	18	20	4	steel	250	none	100,000
11	4	90	40	aluminum	5000	jungle gym, bowling room	10,000,000
12	6	45	12	fiberglass	1200	GPS, lifeboats	1,300,000
13	2	120	60	wood	9000	advanced radar, fire system	14,000,000
14	7	55	18	aluminum	2000	GPS, autopilot	2,800,000
15	10	65	22	fiberglass	2500	safety suite, cinema	3,600,000

In our captain's log, or what you might call a spreadsheet, every column captures a bit of this information. If we had this

information for many ships we could plot them all on a chart, where each ship is a point in a grand, multi-dimensional space. On this chart, the x -axis might show the boat's age, the y -axis its length, and the z -axis the count of its rooms. Add in more details—a w -axis for the quality of materials, a v -axis for the engine power, and even a u -axis for the size of the jungle gym—and we're sailing in higher-dimensional waters.

Even though we can't visualize these higher dimensions, mathematically we can calculate with them and derive meaningful insights from them. Techniques like *dimensional reduction* are used to help simplify these data sets and make them easier to work with. In this way, *dimension* becomes a very flexible term, not just referring to space or time but to any variable that provides us with more information about the system we're examining. Understanding how to make sense of multi-dimensional data could help you in many careers, including those in science, technology, engineering, design, and business.

RETURN OF THE GEMS

Having cataloged the gems, we can start the task of returning them. First on our list is a gem from **Zerosville**, but we don't need to travel to return it. This gem is no ordinary treasure; it's a single point that encompasses the entirety of its universe. Everything that exists in a world of zero dimensions exists within this one point. I gently place this all-encompassing point into my pocket, whispering a promise to keep it safe.

Next is the gem from **Onesville**: a slender, crystalline rod. Onesville, much like Twosville, isn't a place you can find on an ordinary map. Many visitors travel there in their dreams. So, we set sail on the winds of imagination, envisioning

a world of one dimension rising like a long vine from the misty waters before us. It stretches infinitely upward and downward, a line without end.

In Onesville, creatures called Linelings exist along this 1D line, seeing only the rear end of the Lineling in front of them. Linelings are short segments of the line and can only move up and down in a linear fashion.

Calling out to the Linelings, I announce our intention to return their gem, but I can tell straight away that something is wrong. Onesville, usually a chorus of chants, is eerily silent.

On a usual visit to Onesville, each Lineling can be heard proudly declaring their position in the hierarchy of the line. For example, you might hear, "I am 28th in line," followed by "I am 29th in line," followed by "I am 30th in line," and so on. The chanting occurs continuously. At the head of the line, the first Lineling chants, "I am Queen of Onesville." She knows

this because, instead of seeing the back of her fellow Lineling, she sees a glittery gem. She's like the first bead on a necklace.

The line can shift up and down but is infinite. On today's visit, without the chants to tell me what position of the line is in front of us, there's no way of knowing how far away the queen is. This unsettling silence could only mean one thing: Without their gem, the Linelings have lost their sense of order.

I start to climb the line, each Lineling reacting with shock and confusion to my three-dimensional touch, an alien sensation in their linear world. They scream, feeling the sharp pang of my hands and feet inside their segments as I climb.



Realizing the impossibility of my task—to traverse this infinitely long line—I finally decide to place the gem where I am, right in between two random Linelings. No sooner do I do this than the Linelings on either side of the newly placed gem erupt in a chorus of “I am the Queen of Onesville!”

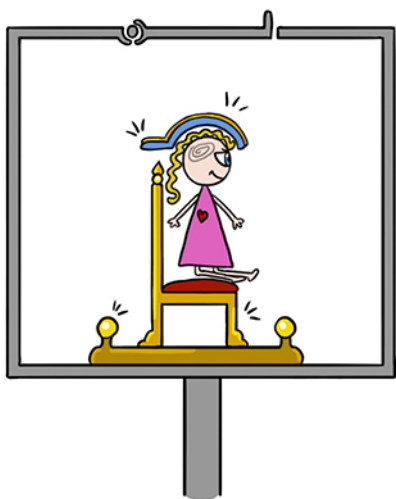
This is unprecedented. In Onesville, one’s place in the line is everything, and suddenly, there are two claiming to be queen. The next two Linelings on the top and bottom both chant, “I am first in line,” and so on down this newly divided line. The Lineling above the gem must have detected her proximity to it despite it being behind her. I was naïve to think that they were limited to perceiving their world through forward vision alone.

The commotion on the line is immediate and intense. The two self-proclaimed queens, now head to head, claim their sovereignty, each challenging the other’s right to the throne. In Onesville, such a challenge is not taken lightly—it’s a sacred matter, and the order of succession is fundamental to their existence.

Chaos ensues. The line, usually a place of strict order and hierarchy, is now a battleground of competing claims and confusion. I hastily retreat, realizing our well-intentioned act has inadvertently caused a war. As we make our way back to our ship, the sounds of upheaval fade into the distance, leaving us with a sense of unease. We set sail toward Twosville, hoping our efforts there will restore balance rather than disrupt it.

Upon returning to the flat expanse where our two-dimensional friends reside, I eagerly peer through my





spyglass, curious about the changes since our last visit. What greets my eyes is a sight beyond my wildest expectations. Our friend Tudy now sits atop a grand throne in the town square, a crown resting atop her head. On closer inspection, the crown is the very hat we flipped, now adorned with a dusting of gold.

Surrounded by a sea of a thousand **Flatlings**, she sits, not just as a queen but as a proponent of mysticism. She narrates her tale of woe and wonder: Once a victim of an invisible thief—a ghost, as she calls it—who whisked away her life's savings, she was left with nothing but despair. Yet, in her darkest hour, she claims that a spirit, summoned by her own special powers, flipped her hat, marking the beginning of her ascent to royalty.

This flipped hat, now a crown, has become proof of her alleged connection with the otherworldly divine. Witnesses testify that the hat is Tudy's original possession, one she has owned for many years, and not merely stolen from a Right-sider.

Tudy's newfound status has led her to advocate for spiritual intervention in times of despair. She offers psychic readings and a lifetime of good luck to those who bring her gifts. This has not only brought her wealth but also shifted the culture of this two-dimensional world toward a belief in the paranormal—a concept previously alien to them.

She doesn't speak of a third dimension; instead, her words paint ghostly images of spirits lurking amongst the

Flatlings, stirring unease and fear. An entire neighborhood, allegedly haunted, stands deserted. More neighborhoods, where items such as socks have reportedly gone missing, are slated for demolition. The Flatlings have embraced these tales wholeheartedly, their lives now shadowed by the fear of nocturnal visits from phantoms that can slip through walls.

This transformation of the community, based on scant evidence, gives me pause. The Flatlings have adopted beliefs and fears that are, to them, beyond verification. It's a stark reminder of the power of untestable claims, especially when only a sliver of information is available. Their readiness to accept these tales as truth, without the means to prove or disprove them, mirrors our own world's penchant for stories that defy explanation.

In our worldly experience we note no shortage of ghost and paranormal stories that involve creatures appearing in closed rooms or vanishing at will. Some try to explain these occurrences as interactions of four-dimensional creatures with our space. As it stands, there is no evidence of a physical fourth spatial dimension. No experiments have proven the existence of such a space, and many have failed trying.

The fourth dimension, as visited in this book, is imaginary. It is a part of mathematical reality, which is more of a psychological space than a physical one. Almost all mathematics is imaginary. There are no perfect triangles, squares, or circles in the physical world.

We don't uncover mathematical reality by placing shapes under a microscope. Instead, we imagine perfect arrangements of shapes and use symmetry, algebra, logic, and language to understand how they work. Onesville, Twosville, and Foursville are mere thought experiments, but so is Threesville. The real world is messier than mathematics can describe.



Let your mind wander back to the imaginary world of Twosville for a moment. Imagine that our 2D friends actually have a teeny bit of thickness, a smidgeon of height that protrudes out into our 3D world. If this small thickness was the same for everything and everyone, and they had no way of sensing or moving in this third dimension, they might not even notice it.

Now, let's stir the pot a little more. If their brains have a wee bit of 3D thickness, could they, perhaps, think in 3D without realizing it? And here's a wild thought:

What if we, ourselves, have a small extension into the fourth dimension, without being aware of it? Unless we collided with something in the fourth dimension, we wouldn't notice it. If the higher-dimensional space is sparse enough for us to move without hindrance, our 4D selves might just be quietly co-existing with us, unbeknownst to our 3D senses.

Ideas like this are fun to think about inside the confines of our imaginary mathematical spaces. While geometrically possible, this doesn't say anything about physical reality. Ideas of such a four-dimensional thickness have been used to try and explain all manner of phenomena, from one's "aura" to soulmates and ghosts.

Here's where the scientific method becomes our guiding light. It's a structured approach to increasing our understanding of the world around us. The method starts with observation—watching, listening, and noting down phenomena. Then comes the hypothesis, an educated guess, a what-if that offers a potential explanation for the observed mysteries.

Next, we roll up our sleeves and dive into experimentation. It's here that hypotheses are put to the test, where we look for tangible, measurable proof. But not all experiments are created equal. The need to design good experiments that can support or negate a hypothesis is crucial to discerning fact from fiction.

A credible experiment is marked by certain key features, such as control variables. These are elements that remain unchanged throughout the test to make sure any observed effects can be confidently linked to the variable we're examining. Also, an experiment should be something others can replicate, using the same steps and conditions to see if they get the same results. This repeatability is a check against fluke findings. Sometimes, experiments even use a double-blind approach, where neither the subject nor the researcher knows certain details, to prevent bias and ensure the results are as honest as can be.

Finally, there's the analysis, where results are examined meticulously and conclusions are drawn. If the evidence is solid, a theory is born. If not, it's back to the drawing board.

We find ourselves at an intriguing crossroads. Can the rigorous scrutiny of the scientific method ever find a spiritual connection to the mathematics of hyperspace? Throughout history, a number of scientists, skeptics, and believers alike have donned their detective hats and tried to do so. However, when we look at their findings, we often see poorly designed experiments, wishful thinking, and even fraud.

If you let your imagination run far enough, you might even be able to explain each of us as no more than the tip of a finger of an infinite-dimensional creature poking its hand into our space. These ideas are fun, but it's essential to remember that, without solid scientific proof, you might

find yourself being taken for a ride by people who have something to gain from your trust.

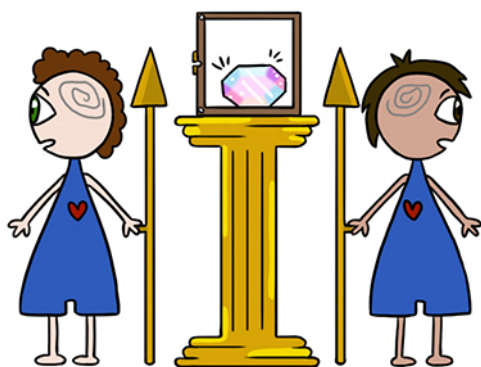
We should remember that stories and anecdotes aren't the same as scientific proof.

Mathematics is fun and interesting, and it can set someone on a path of lifelong learning. However, I have seen first-hand how harm can be done to vulnerable individuals by people who hijack scientific ideas and twist them into something else. When people sell false hope wrapped up in scientific jargon, the harm can be real.

As we witness Tudy reign over her mesmerized Flatlings, I can't help but wonder about the ramifications of our actions and the stories we choose to believe. Our interaction with a simple object has spun a web of beliefs that has transformed an entire society. The two-dimensional world illustrates a fact that is true for us, too: Our understanding of the universe will always be as thin as a piece of paper in the heavy book of the truth.

The locked safe of Queen Tudy rests on a pillar near her throne. With precision, I throw the stolen gem overboard so that it lands back in the closed box. For a fleeting moment, I feel a sense of relief, as if our guilt is washing away. However, this serenity is short-lived.

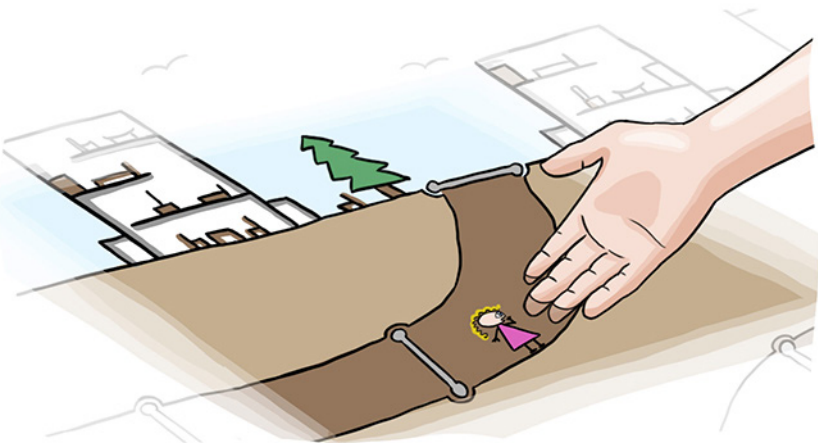
The sound of the gem landing back in the box, clattering against the walls, startles the queen's guards. With spears raised, they gather around and unlock the safe with a key. Inside they find the



returned gem. Grasping for an explanation, the guards latch on to what seems like the simplest explanation before them, that their queen is practicing witchcraft. Word spreads through the castle and through the town. The Flatlings, already on edge about ghosts and hauntings, come for their queen.

Tudy is chased from her castle, her crown tumbling from her head as she flees. She runs on and on, never looking back. Narrowly escaping with her life, she seeks refuge in a tunnel beneath the desolate ghost town. Some relentless Flatlings continue their search for her. Meanwhile, a new successor, seizing the opportunity, picks up her discarded crown and assumes the throne, perpetuating tales of suspicion to cement their rule.

Let's stop at the tunnel and speak to Tudy. Upon seeing the impact of my actions, I would like to right my wrongs. To make amends with Tudy, I offer her one final gift: not a physical object from our dimension, but the gift of knowledge. I invite her to join us on a tour through the dimensions, a chance to experience the wonders of the universe beyond her flat realm. This adventure is also an opportunity for her to gain insights and wisdom that she



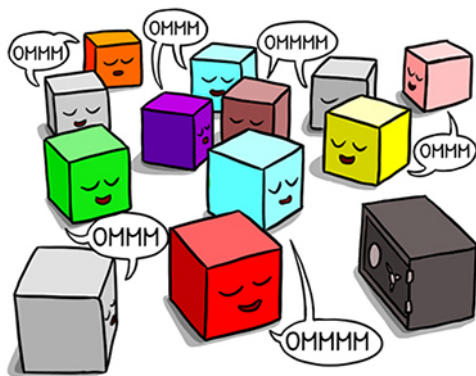
can take back to her people, thereby enriching their world in ways far more valuable than any gem or hat.

Tudy is delicate, like a paper cutout, her perception limited to mere slices of our three-dimensional world. As we transport her, she scans our surroundings like a barcode reader, slowly piecing together the vastness of a realm she's never imagined. I hand her a bucket, just in case she gets nauseous trying to stitch together these complex visuals.

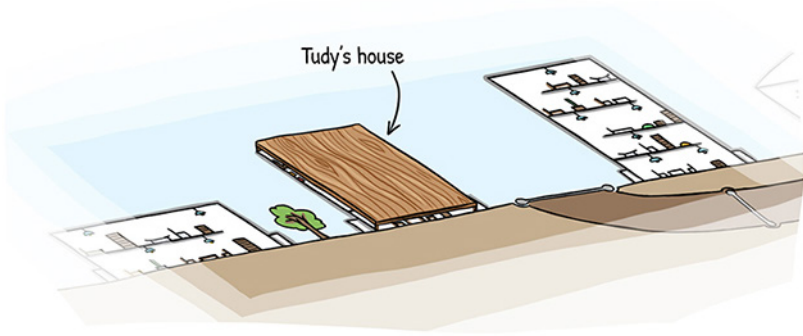
First, we show Tudy her own world from a vantage point she never knew existed. Twosville's underground mines, buried treasures, and even inside her neighbor's house. Then, we show her our world. The mountains, the sky, birds flying, balloons floating, the sun setting.

As we're showing Tudy the third dimension, we cross paths with Hinton's cubes, now leaders of a meditation school. They claim that, if someone spends long enough watching the cubes dance, that individual will achieve enlightenment and be able to visualize higher-dimensional shapes.

Hinton's cubes have gained quite the cult following, but when pressed, they refuse to answer the question "So what does a hypersphere look like?" Nevertheless, we discreetly deliver the gems they so coveted while they meditate. At least their new life is less troubling than their time as pirates.



To make amends for the turmoil caused in Onesville and Twosville, I gather my crew to take on a project of goodwill: the reconstruction of Tudy's house. We fortify it with the strongest doors and locks. As a courtesy, we also add a feature no other house in Twosville has: a thin sheet of wood providing a veil of privacy against prying eyes from above. Her home now slightly protrudes into the third dimension by a tenth of a millimeter.



I also prepare a speech to deliver to the inhabitants of Twosville, explaining the truth behind the flipped hat and the stolen gems. It's important that they understand the nature of these objects and the inadvertent role they played in altering the social order of their world.

Anchoring our ship just above their plane, I lean in, my face appearing to them as a series of shifting outlines, much like slices of an MRI scan. Some of the more skeptical Flatlings launch spears at my face, an understandable reaction given their recent upheaval. Fortunately, my crew, hovering above their two-dimensional world, catch these spears, pulling them out of the 2D plane.

I deliver my speech and, overall, it is well received. Once their supply of spears runs out, the Flatlings decide that they'd better listen to me or face grave danger. I implore

them to respect Tudy, who returns and is now enlightened by her interdimensional travels.

She embraces her newfound knowledge, starting a YouTube channel where she makes short videos about dimensions. She draws sketches of a one-dimensional character called Onedy. She uses the interaction with this



1D character as an analogy to teach Flatlings about the existence of the third and higher dimensions. She captivates an audience with lessons

where she says things such as, "A 3D creature could appear in a closed room, take the yolk from an egg without breaking the shell, and see inside a Flatling's body."

Tudy writes a book of her teachings and calls it *A Guide to Making Friends in the Third Dimension*. In it, she says that 3D creatures can be scary, friendly, big, small, well-meaning, or just mean. Upon reading her book I was flattered to be referred to as a friend. One sentiment from her book really resonated with me: While discussing the return of her gems, she said that the real treasure was the math she learned along the way.

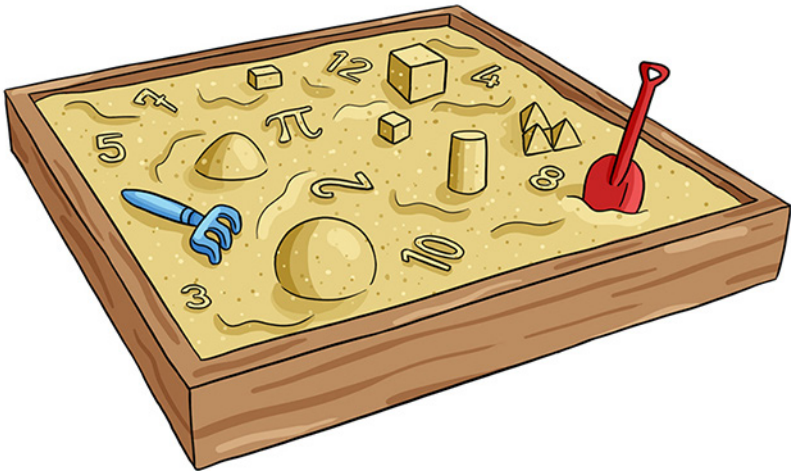
BACK TO SHORE

As our ship gracefully glides back toward the shore, the time has come for us to part ways, at least for now. I am tasked with making amends. Armed with the remainder of the stolen gems, I am off to venture across dimensions once more to return these treasures to their rightful owners in the fifth, sixth, and higher dimensions. I hope that those friends will accept me as kindly as you did.

I hope you had fun today. Please remember that this doesn't need to be the end of your dimensional journey. In fact, I like to believe it's just beginning. Our escapade through the second, third, and fourth dimensions was just a glimpse. If math is to be trusted, there is an infinitude of dimensions that can be explored. These are not necessarily physical, and you might never be able to set foot in them, but you can play there in a mathematical playground.

To me, math is playing in the sandpit of numbers and seeing what you can build. I'm just another kid in the sandpit who whispered in your ear, "Hey, have you ever thought about building a shape like this?" while pointing at a sandy tesseract. Your own sandcastle tesseract might have existed in your mind for only a second before crumbling, or it might not have existed at all, but at least you tried to build more than just a mountain of sand.

Go *forth*, and see what else you can build. I will be waiting excitedly to see what you create.



LOOKING OUT

Look out at the horizon,
At the world you have eyes in.

If you had an eye inside your brain,
In 4D, there's another view you'd gain.

An unobstructed view, ana, from inside your skull,
And all the way kata, in case ana is dull.

But that whole scene is just a scrap of paper;
In 5D, the blueprints of a skyscraper.

Stray from the plans, it doesn't matter;
To someone in 6D, it couldn't seem flatter.

The patterns of mathematics repeat,
Even if our grasp is incomplete.

That glowing n -ball is the setting sun,
When it rises, it might be $n + 1$.

ACKNOWLEDGEMENTS

Writing this book felt like diving into the deep end. For more than two years, I was swirling around in interdimensional waters. Thank you to the following people, without whom I just might have drowned.

Martina Pepiciello, who was the perfect person to illustrate this book. The illustrations are so important to this book and to communicating complex ideas clearly. Martina was a joy to work with and, with her physics expertise, was able to make sense of even my strangest sketch ideas.

CodeParade, who kindly contributed his ideas to “The Games Room” chapter and provided valuable feedback on this book’s trip to Foursville. As well as running a YouTube channel, CodeParade is the mastermind behind the games *4D Golf* and *Hyperbolica*, both of which have helped to tune my ability to think about higher-dimensional or non-euclidean settings.

Michael Stevens, for writing the foreword. There are few people I know as passionate about the fourth dimension as him. Michael has continually inspired me in the pursuit of communicating ideas and embracing unusual aspects of our world. Hearing about his own attempts to visualize a hypersphere stuck with me and pushed me to try harder to think about such shapes. Michael makes fascinating and thoughtful videos on his YouTube channel *Vsauce*.

Scott Garrett, for creating the cover artwork. As a high school student, I remember buying myself my first science book; it was a copy of *We Need to Talk about Kelvin* by Marcus Chown. That book is special to me, and so how special is it that I managed to get Scott, the cover illustrator of Chown’s book, to do my own cover? Scott was amazing to work with and came up with a cover I love.

Henry Segerman, for answering my questions about the Hopf fibration and for providing feedback on the “Box Jellyfish” chapter.

Melanie Madden, for editing the book and making suggestions that helped to keep the ideas as clear as possible.

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Lastly, my friends and family for keeping me afloat.

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The Vessel of Higher Dimensions

■ *Sanome*

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Underwater Rugby ■ *Stephen John Codd*

Universe Loaf ■ *Mike Melchior*

2D World ■ *Michael Lytek*

Flatling House ■ *Nicolas*

Town Square ■ *Eric Floehr*

Submarines ■ *John Douglas Trask*

Anatomy ■ *David Yeargin*

Puppet Show ■ *Raz Tirosh*

Perspective Shift ■ *Tales Ribeiro*

Flatling Vision ■ *Ella P*

Gem in Safe ■ *Robin P.*

Flipping Hat ■ *Bill Kyle*

Our Impossible Gift ■ *neal k williams*

Stealing a Gem ■ *Alfredo Rosales*

Snorkeling ■ *Raz Tirosh*

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Hopf Fibration ■ *Fuat Baran*

Hiding ■ *Daniel Brosseau*

Rubik's Cube ■ *Steve Chantry-Taylor*

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Hyperpeach Tree ■ *R. Arulambalam*

Caterpillar ■ *Roland Soliz*

4D Person ■ *Rob Westwood*

Box of Hypercubes ■ *Bryan Williams*

4D Wheel ■ *Fabian*

Conic Sections ■ *Anakin Hamilton*

Ant on Orange ■ *Sam Asplin*

Dodgeball ■ *Rick Blaine*

4D Slide ■ *Norbert van Woerkom*

It's a Flat World ■ *Marcus von Cube*

Picasso ■ *John Drake*

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4D Dice ■ *Koen Van Bortel*

Australian Coastline ■ *Adam*

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Dolphin Tricks 2 ■ *Gayathri T. Sri*

Dolphin Tricks 3 ■ *John Bosco*

Fundamental Forces ■ *Scott Martin*

Particle Accelerator ■ *Dominic Lagrois, Ph.D.*

String Theory ■ *ketjap maniz*

7D Cube ■ *Cesar Barria*

FURTHER READING/WATCHING/PLAYING



Abbott, Edwin A. (1884). *Flatland: A Romance of Many Dimensions*.

Flatland is the first book I ever read about the idea of the fourth dimension. It has inspired many parts of this book but most notably the use of a two-dimensional land to set up an analogy for how 4D creatures may view a 3D land. In *Flatland*, the Flatlanders are shapes which slide around on their infinite plane like air hockey pucks. This differs from my fictional world of Twosville, where there is a ground and sky orienting the Flatlings, and a force of gravity pulling them toward the center of their circular planet.



Hinton, Charles. (1888). *A New Era of Thought*.

Swan Sonnenschein & Co. Ltd.

Hinton was another early pioneer of popularizing ideas about the fourth dimension. It was in *A New Era of Thought* that Hinton first coined the term *tesseract* to describe a four-dimensional cube. He also introduced the terms *ana* and *kata* for the unknown extra directions.



Hinton, Charles. (1906). *The Fourth Dimension*.

Swan Sonnenschein & Co. Ltd.

In *The Fourth Dimension*, Hinton further develops his system of colored cubes to try and model four-dimensional perception. The book was printed with a colored fold-out showing the cubes, and some readers received a physical box of wooden cubes.



Bragdon, Claude. (1913). *A Primer of Higher Space*.

The Manas Press.

A Primer of Higher Space taught me a few ways to draw a tesseract and inspired the illustration of the box full of 4D dice in Part 4 of this book.



Rucker, Rudy. (1977). *Geometry, Relativity, and the Fourth*

Dimension. Dover Publications.

Geometry, Relativity, and the Fourth Dimension is a really lovely book that covers many of the same ideas that I have presented here. Rucker does a great job of connecting the idea of a fourth spatial dimension to spacetime and wider ideas of philosophy and even spirituality.



Dewdney, A. K. (1984). *The Planiverse: Computer Contact with a Two-Dimensional World*. Poseidon Press.

The Planiverse takes a very in-depth look at a hypothetical 2D world. It inspired some aspects of Twosville, such as what the Flatling's cars might look like.



Gardner, Martin. (1964). *The Ambidextrous Universe: Left, Right, and the Fall of Parity*. Penguin Books.

This book covers chirality, handedness, and all the interesting ways that symmetry is a part of our world. I especially enjoyed its coverage of the parity-breaking experiments of Chien-Shiung Wu.



McMullen, Chris. (2008). *The Visual Guide to Extra Dimensions, (Volumes 1–2)*. Custom Books.

Volume 1 of *The Visual Guide to Extra Dimensions* introduced me to the name *glome* for a 4D sphere. It also covers the full complexity of 4D shapes. Volume 2 speaks to modern physics experiments, including a good explanation of Kaluza-Klein excitations.



Scargill, J. H. C. (2020). "Existence of life in 2+1 dimensions." *Physical Review Research*, 2(1), 013217.

<https://doi.org/10.1103/physrevresearch.2.013217>

A scientific look at the possibility of life in other dimensions, including mention of atoms and orbits.



Arkani-Hamed, N., Dimopoulos, S., & Dvali, G. (1998). "The hierarchy problem and new dimensions at a millimeter." *Physics Letters B*, 429(3-4), 263-272.

This paper informed some of my "Theory of Everything" chapter in Part 5. It discusses possible explanations for why gravity appears much weaker than the other fundamental forces.



Adamala et al. (2024). *Technical Report on Mirror Bacteria: Feasibility and Risks*. Stanford Digital Repository.

<https://doi.org/10.25740/cv716pj4036>

This report is a lengthy review of the dangers of creating mirror life. It motivated the decontamination scene as we left Foursville.



Johnson, Niles. (2011). "A visualization of the Hopf fibration."

<https://nilesjohnson.net/hopf.html>

Included on this webpage is an amazing animation of the Hopf fibration. It shows various circles and their corresponding point on a sphere. The animation displays the stereographic projection in a way that is difficult to put into words. The rest of the page is also helpful in trying to understand the concept.



Ten Bosch, Marc. (2017). *4D Toys*. <https://4dtoys.com>

In this game, you can roll 4D dice, play with 4D shapes, and even take a look at a Hopf fibration model. This game proved very useful in visualizing some Foursville scenes.



CodeParade. (2024). *4D Golf*.

https://store.steampowered.com/app/2147950/4D_Golf/

At the end of our visit to Foursville, we saw a mini-golf course. That was a mini-taster of the mind-bending ideas on display in the game *4D Golf* by CodeParade. *4D Golf* offers a hands-on experience of the fourth dimension and beyond. I highly recommend it to anyone who likes clever puzzles or who just wants to imagine they live in a 4D world.



Videos documenting the game development process as well as other mathematical ideas such as hyperbolic space can be found on CodeParade's YouTube channel: youtube.com/@CodeParade

GLOSSARY

0D: zero-dimensional (i.e., a point)

1D: one-dimensional (i.e., a line)

2D: two-dimensional (e.g., a square)

3D: three-dimensional (e.g., a cube)

4D: four-dimensional (e.g., a tesseract)

Ana: one of the six cardinal directions in four-dimensional space: north, south, east, west, ana, kata. It comes from the Greek word for “up toward.”

Branes: higher-dimensional objects proposed in some advanced physics models, acting like extended “surfaces” beyond the idea of one-dimensional strings.

Chiral: a property of an object that is distinguishable from its mirror image.

Compact dimensions: additional directions in certain physical models that are curled up at very small scales, making them difficult to detect in everyday experiments.

Cross-section: the shape exposed by making a complete, straight slice through something.

Cubinder: a shape in four-dimensional space created by “extruding” a cube in one extra dimension. You can think of it like a 4D version of a regular cylinder, except you start with a cube instead of a circle.

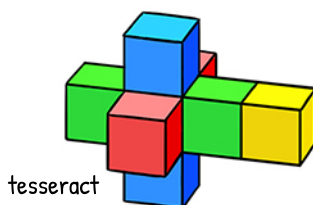
Dark matter: a hypothetical substance that reveals its presence through the gravitational pull it exerts on stars and galaxies while remaining undetectable by ordinary light-based instruments.

Dimension: Unless otherwise stated, *dimension* refers to spatial dimension, which is the number of degrees of freedom to move within space. (North-south, west-east, and up-down are the three degrees of freedom we have in three-dimensional space.)

Disk: a filled-in circle; a two-dimensional shape that has an area.

Electromagnetic force: the interaction responsible for electric and magnetic effects, such as light and magnetism, acting on charged particles.

Electroweak force: a unified force model combining the electromagnetic interaction and the weak interaction.

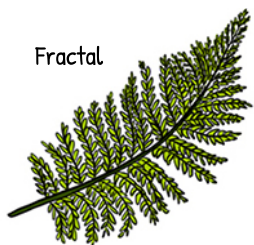


Ellipse: a smooth, oval-like curve. You can think of it as all the points for which the total distance to two fixed points (called *foci*) always adds up to the same value. A three-dimensional analog is called an *ellipsoid*.

Fibration: a concept in topology that describes a way of mapping one space onto another, where each point in the “base” space has a corresponding “fiber” that looks roughly the same, no matter which point you choose.

Flatling: a fictional word for a two-dimensional creature.

Fractal



Fractal: a shape that contains detail at every scale (even when you zoom in close).

Foursville: a fictional word for a four-dimensional world.

Fourth dimension: a theoretical realm where there are four dimensions of space: north-south, east-west, up-down, and ana-kata.

Glome: Also called a 3-sphere or 4D hypersphere, this is an analog of the sphere that exists in 4D space. It is a three-dimensional boundary, where all points lie the same distance from a single center.

Glomitude: coordinates that help describe positions in a four-dimensional round shape, extending the idea of latitude and longitude into an extra dimension.

Gongol: the name given to a hyperball in four-dimensional space. This four-dimensional solid region includes all points within a specific distance from a center—akin to the filled interior of a sphere, but one dimension higher.

Grand Unification Theory (GUT): a theoretical model aiming to merge the strong, weak, and electromagnetic interactions into a single force at very high energy levels.

Graviton: a hypothetical elementary particle that gives rise to the gravitational field.

Gravitational force: a universal attraction acting between all objects that have mass, pulling them toward one another.

Handedness: See *Chiral*.

Hilbert space: a mathematical setting used in quantum mechanics to describe all possible states of a system.

Hopf fibration: a famous example of a fibration discovered by Heinz Hopf. It shows how a three-dimensional shape called a 3-sphere (that exists in four-dimensional space) can be mapped onto an ordinary 2-sphere (such as the 2D surface of a ball), with each point on the 2-sphere corresponding to a circle in the 3-sphere.

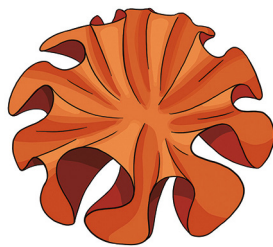
Hyper: over, beyond, or above.

Hyperball: a version of a ball that extends into higher dimensions, containing every point within a certain distance from a center.

Hyperbola: a curve composed of two separate mirror-image arcs. For any point on one of those arcs, if you measure its distance to two fixed points (also called *foci*), the difference between those distances is always the same. A three-dimensional analog is called a *hyperboloid*.

Hyperbolic geometry: a type of geometry where the standard “flat” rules of Euclidean geometry do not apply. It has constant negative curvature, and straight lines behave differently than they do on a flat plane.

Hyperbolic surface: a two-dimensional instance of a space where hyperbolic geometry applies. Every point on the surface is a negatively curved saddle shape, even if the overall shape of the surface is more complex.



Hyperbolic
surface

Hypercube: a shape analogous to a cube that exists in a space with more than three dimensions.

Hyperspace: space with more than three dimensions.

Hypersphere: a shape analogous to a sphere that exists in a space with more than three dimensions. It is the set of points a fixed distance away from a central point.

Hypersurface: a generalization of a surface to higher dimensions. In a space with n dimensions, a hypersurface is an $(n-1)$ -dimensional subset.

Hypervolume: an extension of the familiar volume measurement into spaces with more than three dimensions.

Hypotenuse: the longest side in a right triangle, the one opposite the right angle.

Kaluza-Klein excitations: predicted energy levels or modes that arise when extra dimensions are included in a theory, giving hidden directions that can alter how particles behave.

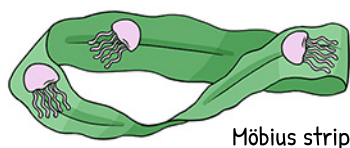
Kata: one of the six cardinal directions in four-dimensional space: north, south, east, west, ana, kata. It comes from the Greek word for “down from.”

Latitude: a way to measure how far north or south a location is on a planet's surface, typically starting at the equator.

Linear: extending along a straight line. A linear equation makes a straight line when it is plotted on a graph.

Longitude: a way to measure how far east or west a location is on a planet's surface, typically referenced from a prime meridian.

Mirror symmetry: when an object can be reflected (such as in a mirror or along a line) and remain unchanged.



Möbius strip: a one-sided surface that can be made by adding a single 180-degree twist to a rectangle before joining its two ends.

n -ball: all the points in n -dimensional space that lie at or within a certain distance from a center.

n -sphere: all the points in n -dimensional space that are equidistant from a central point.

Onesville: a fictional word for a one-dimensional world (an infinite line).

Parabola: a U-shaped curve where every point is equally distant from a fixed point (called the *focus*) and a straight line (called the *directrix*). The path of a projectile under the influence of gravity follows a parabola. A three-dimensional analog is called a *paraboloid*.

Pringle: a potato chip that has a saddle shape, curving away from a central point in two distinct directions; an example of *hyperbolic geometry*.

Qubit: Short for *quantum bit*, qubits are the basic units of information in quantum computing. Unlike regular bits (which are either 0 or 1), qubits can be both 0 and 1 at the same time (a property known as *superposition*).

Radius: The radius of a circle (or sphere) is the distance from its center to any point on its boundary (or surface).

Space-filling fractal: a special repeating pattern that can cover an entire region without leaving any gaps.

Spacetime: a four-dimensional continuum that combines three dimensions of space and one dimension of time.

Sphere: in three-dimensional space, the set of points a fixed distance away from a central point. Unless otherwise stated, *sphere* in this book—and often in mathematics—refers to a hollow sphere, sometimes thought of as the surface of a sphere. A solid sphere is known as a *ball*.

Spherinder: Sometimes called a *spherical cylinder*, this is another four-dimensional shape. It's formed by moving a 3D sphere through a new dimension, much like a cylinder is formed by moving a 2D circle along a line.

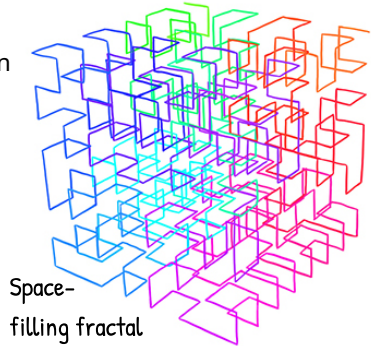
Squaring: Squaring a number means multiplying it by itself. For example, $3^2 = 3 \times 3 = 9$

String theory: a theory that fundamental particles are tiny vibrating “strings,” with different vibration patterns corresponding to different types of particles.

Strong nuclear force: the interaction that binds protons and neutrons in an atom's nucleus and holds quarks together inside those particles.

Tesseract: a four-dimensional hypercube.

TesserComm: a fictional device that can be used to communicate with other dimensions.



Theory of Everything (TOE): a hypothetical complete description that unifies all known physical phenomena, including gravity, into one overarching framework.

Threesville: a fictional word for a three-dimensional world where perfect mathematical shapes can exist (e.g., a perfect sphere).

Torus (plural tori): the shape of a doughnut or a bagel, with a hole in the middle and a ring-like surface. In more formal terms, it's a surface formed by revolving a circle around an axis that is in the same plane but does not intersect the circle.

Twosville: a fictional word for a two-dimensional world. Twosville exists on the surface of an imaginary lake. It is a plane separating the 3D world's air and water.



Variables

Variables: symbols (like x or y) used in math and science to represent numbers or values that can change.

Weak nuclear force: the interaction that causes certain types of radioactive decay and allows particles to change identities (such as turning a neutron into a proton).

Wireframe: a visual representation of an object where only the edges are shown.

Worldline: the path an object follows through spacetime, showing its location at each moment in time.

Wormhole: a speculative tunnel-like feature in spacetime that could, in principle, connect two distant points or regions.

Zerosville: a fictional word for a zero-dimensional world.

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